Regression methods on the cone of positive-definite matrices

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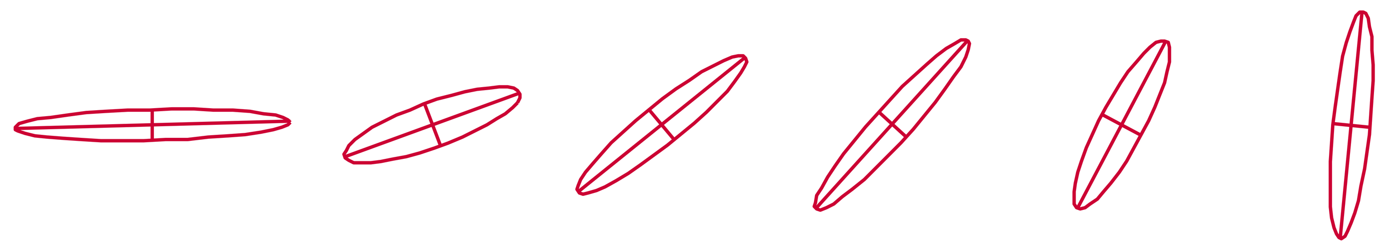
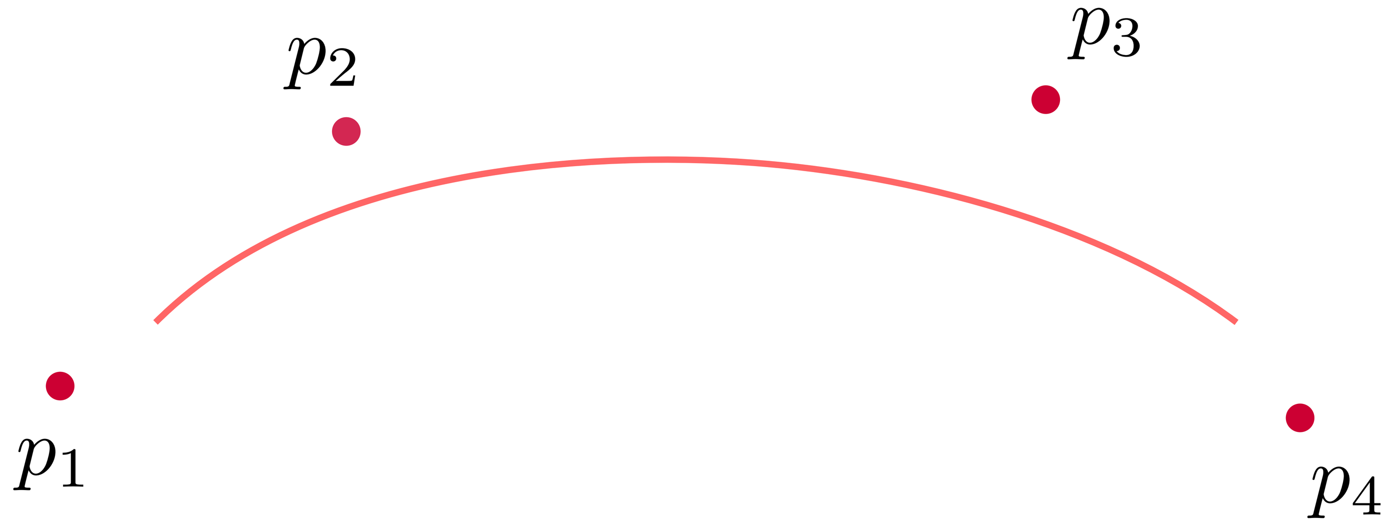
We filter sampled signals whose values are positive-definite matrices.



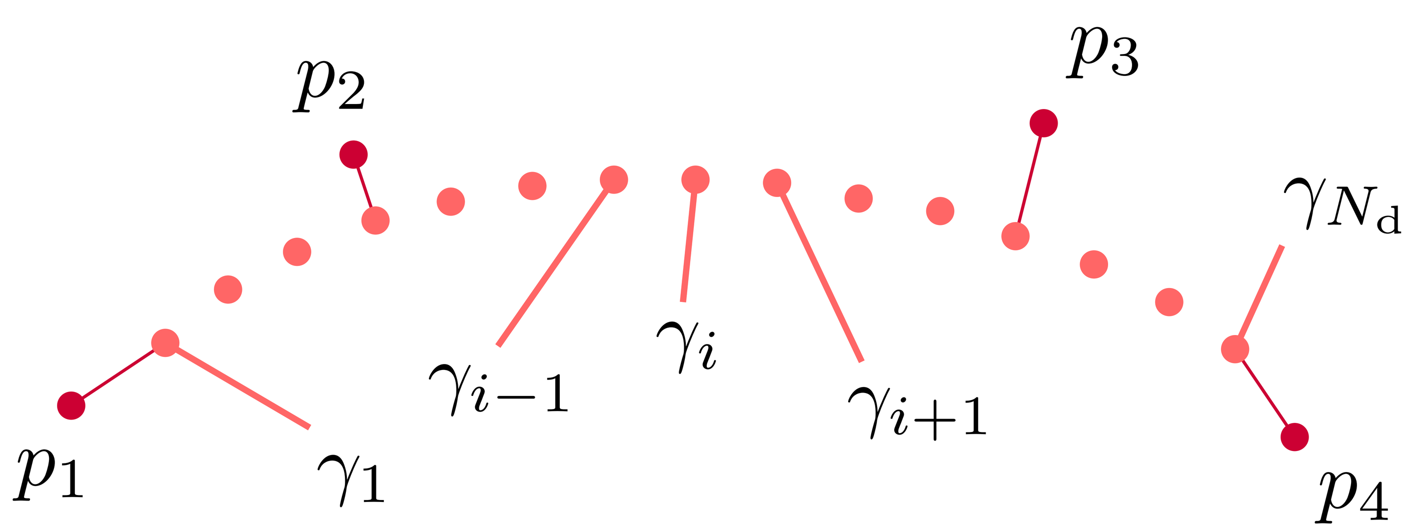
Example: the signal could be the correlation matrix of a dynamic stochastic process.

In a vector space, filtering (or regression) is often cast as an optimization problem for either continuous or discrete curves.



The cost function is a penalty on misfit, speed and acceleration.

$$\hat{E}(\gamma) = \boxed{\sum_{i=1}^{N} \|p_i - \gamma(t_i)\|^2} + \lambda \boxed{\int_{t_1}^{t_N} \|\dot{\gamma}(t)\|^2 \mathrm{d}t} + \mu \boxed{\int_{t_1}^{t_N} \|\ddot{\gamma}(t)\|^2 \mathrm{d}t}$$



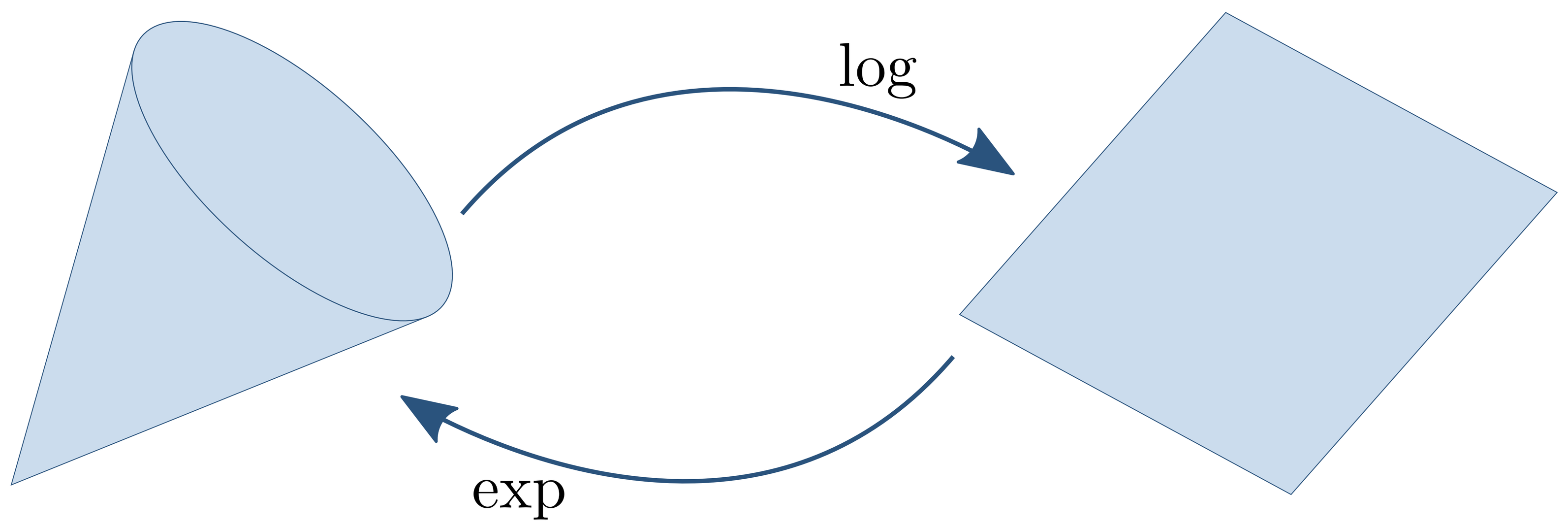
$$E(\gamma) = \boxed{\sum_{i=1}^{N} \|p_i - \gamma_{s_i}\|^2} + \lambda \boxed{\sum_{i=1}^{N_\mathrm{d}-1} \alpha_i \left\| \frac{\gamma_{i+1} - \gamma_i}{\Delta\tau} \right\|^2}$$
$$+ \mu \boxed{\sum_{i=2}^{N_\mathrm{d}-1} \beta_i \left\| \frac{(\gamma_{i+1} - \gamma_i) + (\gamma_{i-1} - \gamma_i)}{\Delta\tau^2} \right\|^2}$$

This is simply a linear least squares objective.

But the cone of positive-definite matrices is not a vector space, hence there is a need for a broader framework.

We investigate two methods based on two different Riemannian geometries.

Method 1: Use the matrix logarithm to map the problem back to the set of symmetric matrices, which is a vector space.



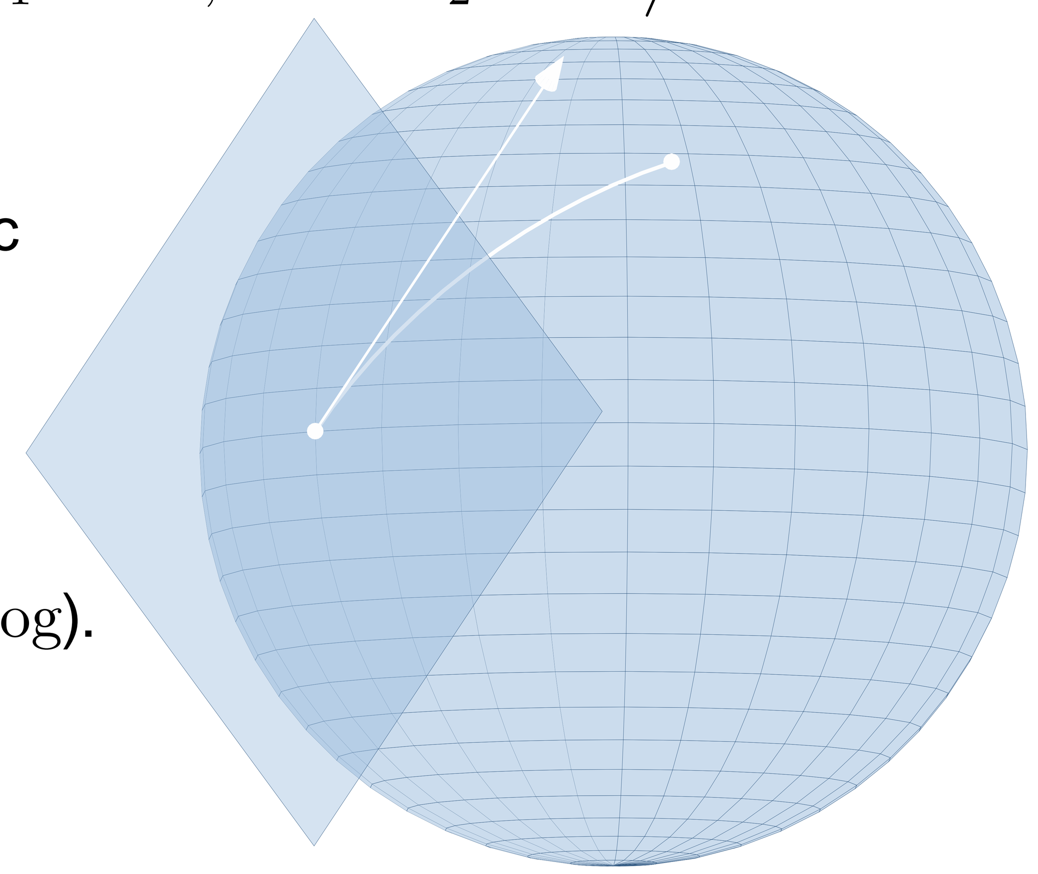
Method 2: Explicitly fit the curve on a manifold by generalizing the objective function *E* and minimizing it.

0. Define a Riemannian metric on the cone:

$$\langle H_1, H_2 \rangle_A = \left\langle A^{-1/2} H_1 A^{-1/2}, A^{-1/2} H_2 A^{-1/2} \right\rangle$$

1. Replace Euclidean distances with geodesic distances.

2. Generalize finite differences with the logarithmic mapping (Log).



$$\mathrm{Log}_x(y) \approx y - x$$

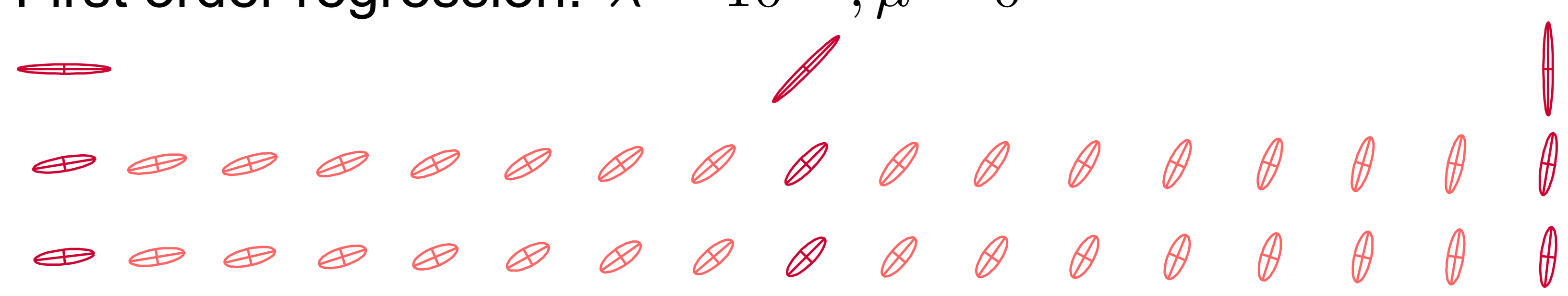
$$E(\gamma) = \boxed{\sum_{i=1}^{N} \mathrm{dist}^2(p_i, \gamma_{s_i})} + \lambda \boxed{\sum_{i=1}^{N_\mathrm{d}-1} \alpha_i \left\| \frac{\mathrm{Log}_{\gamma_i}(\gamma_{i+1})}{\Delta\tau} \right\|^2}$$
$$+ \mu \boxed{\sum_{i=2}^{N_\mathrm{d}-1} \beta_i \left\| \frac{\mathrm{Log}_{\gamma_i}(\gamma_{i+1}) + \mathrm{Log}_{\gamma_i}(\gamma_{i-1})}{\Delta\tau^2} \right\|^2}$$

We need to minimize this objective under the constraints that the points γ_i belong to the manifold of positive-definite matrices. To this end, we use feasible optimization methods.

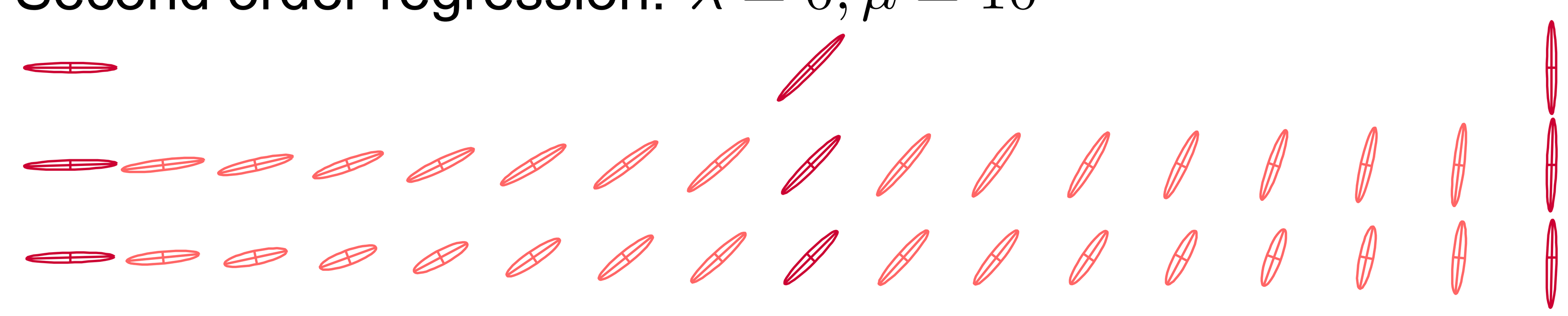
Results are very similar but computation times and complexity differ greatly.

We show regression curves across three data points for various parameter values and with method 1 (first line) and method 2 (second line)

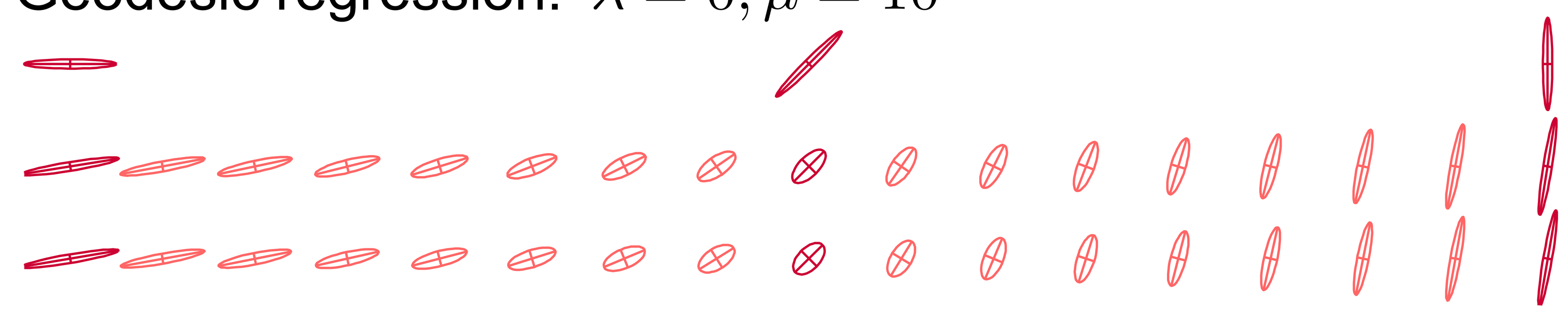
First order regression: $\lambda = 10^{-1}, \mu = 0$



Second order regression: $\lambda = 0, \mu = 10^{-3}$



Geodesic regression: $\lambda = 0, \mu = 10^{3}$



Both methods are efficient for first order regression, but the second method may need many iterations to reach convergence for second order regression.

[1] N. Boumal, P.-A. Absil, *Regression methods on the cone of positive-definite matrices*, In Proceedings of ICASSP 2011, Prague.

[2] N. Boumal, P.-A. Absil, *A discrete regression method on manifolds and its application to data on SO(n)*, In Proceedings of IFAC 2011, Milan.

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