Cramér-Rao bounds
for synchronisation of rotations

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Joint work with A. Singer, P.-A. Absil and V.D. Blondel

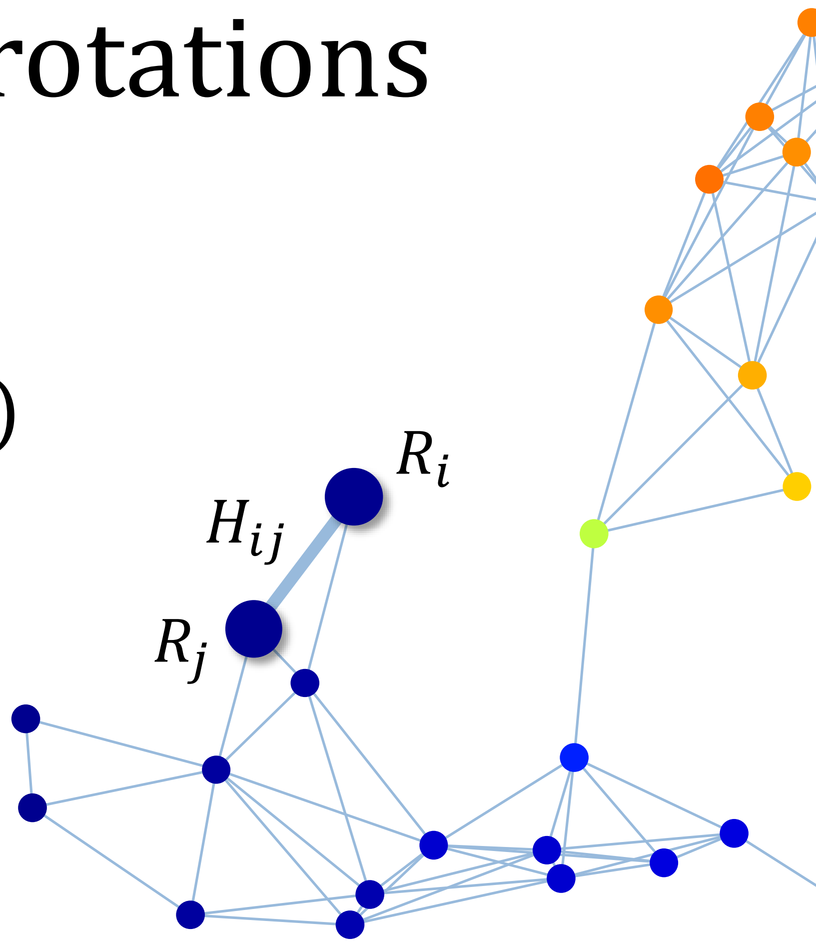
Synchronisation of rotations

Rotations to estimate:

$$R_1, R_2, \ldots, R_N \in \mathrm{SO}(n)$$

Measurements:

$$H_{ij} \approx R_i R_j^{-1}$$

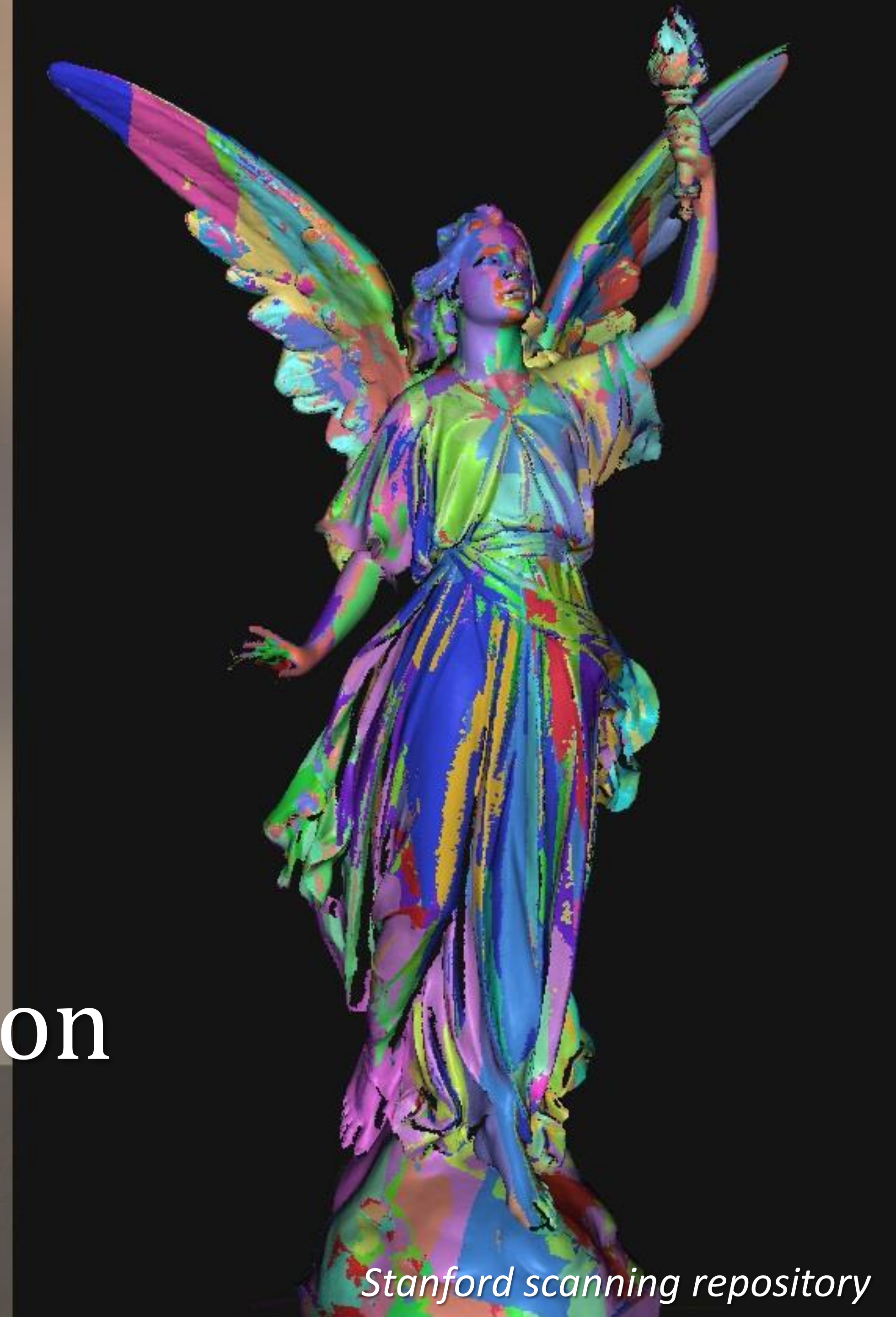


$$\mathrm{SO}(n) = \{R \in \mathbb{R}^{n\times n} : R^T R = I \text{ and } \det R = +1\}$$

Structure from motion

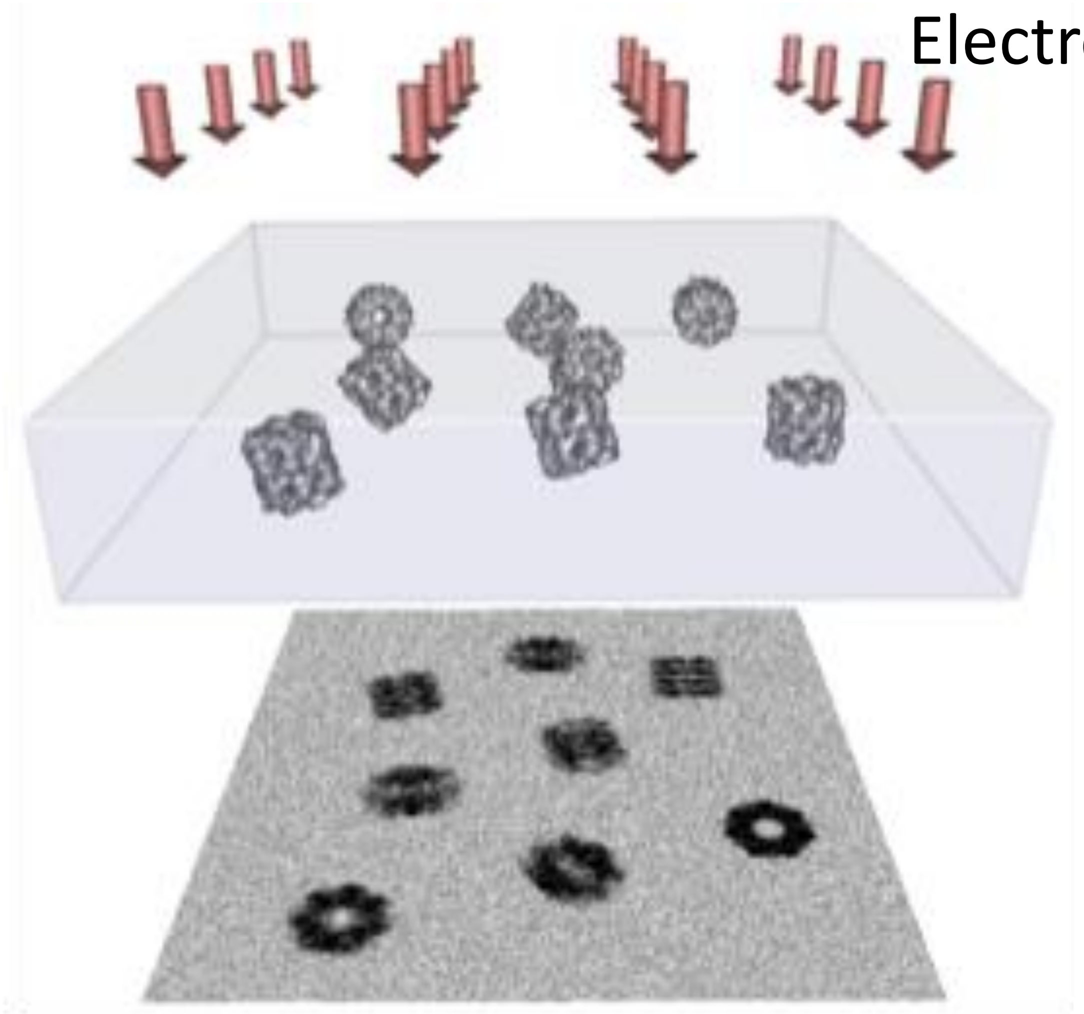
Bundler: Structure from Motion (SfM), Noah Snavely

3D scan registration



Stanford scanning repository

Cryo-Electron Microscopy



https://people.csail.mit.edu/gdp/cryoem.html

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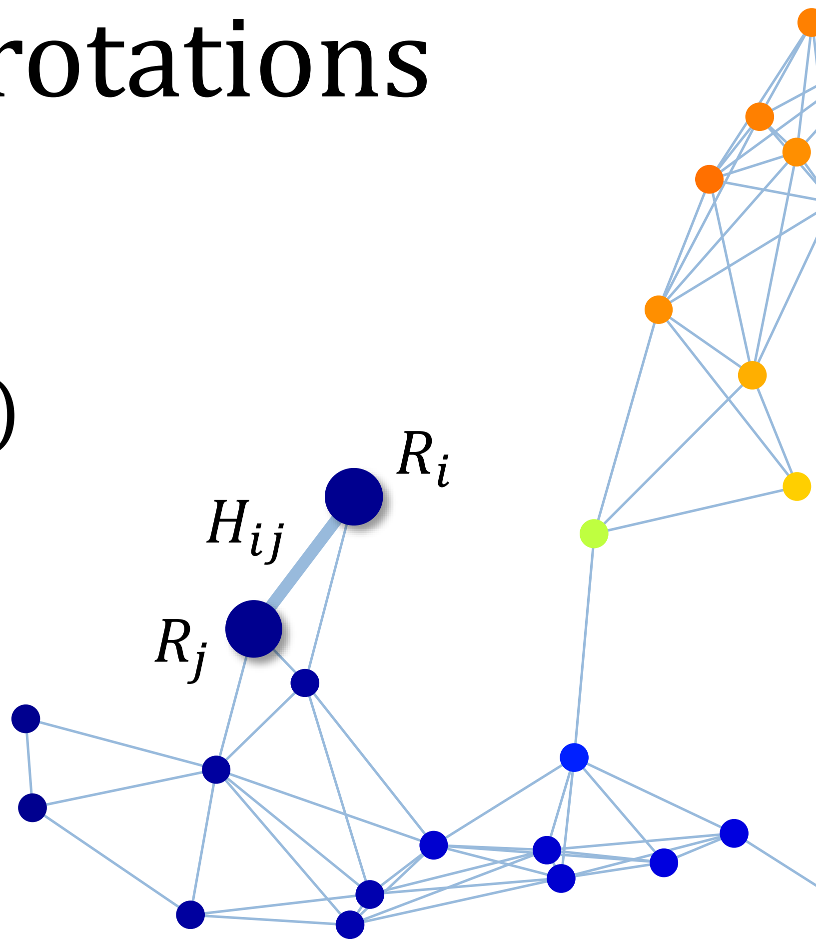

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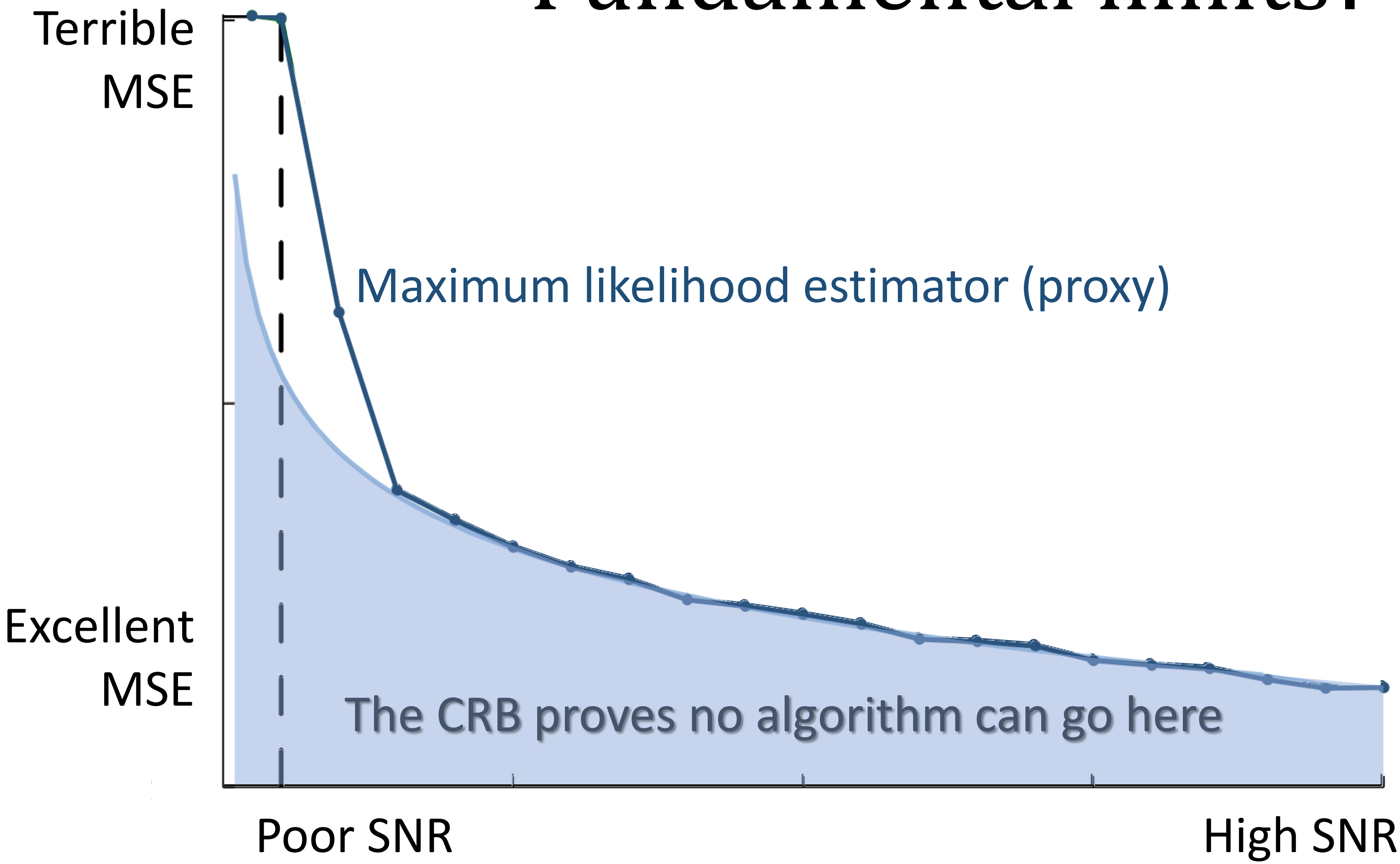
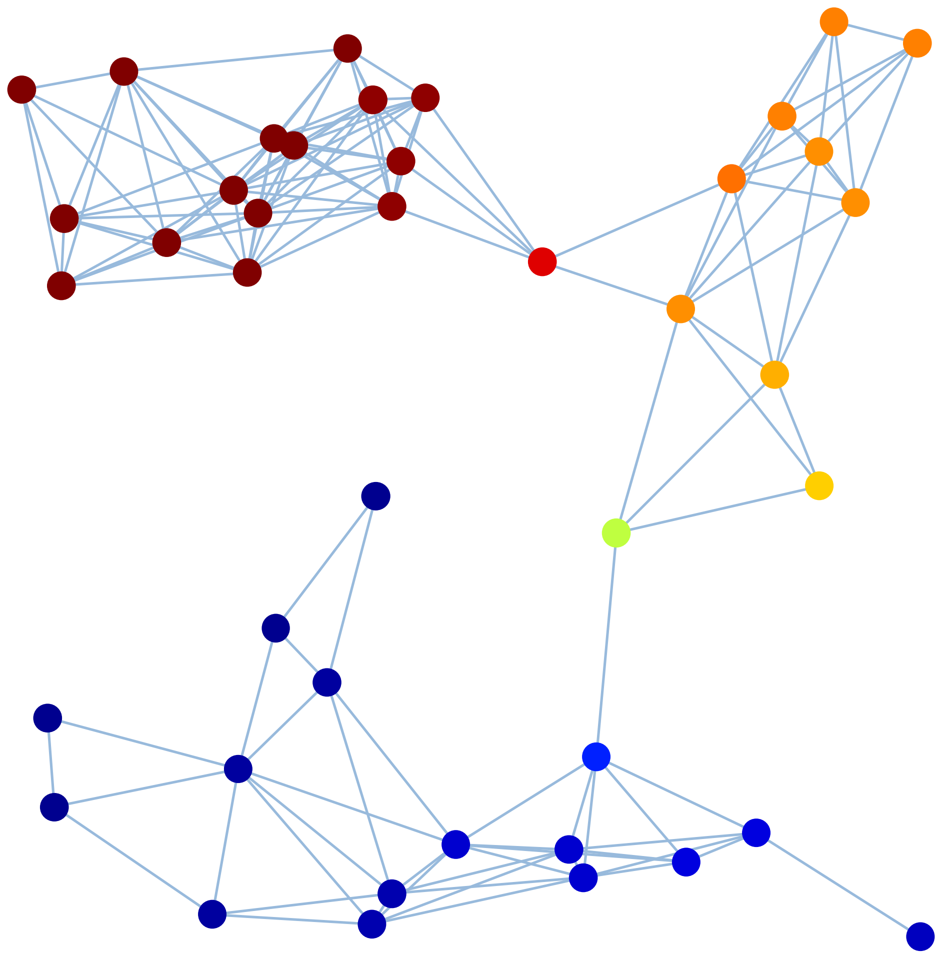
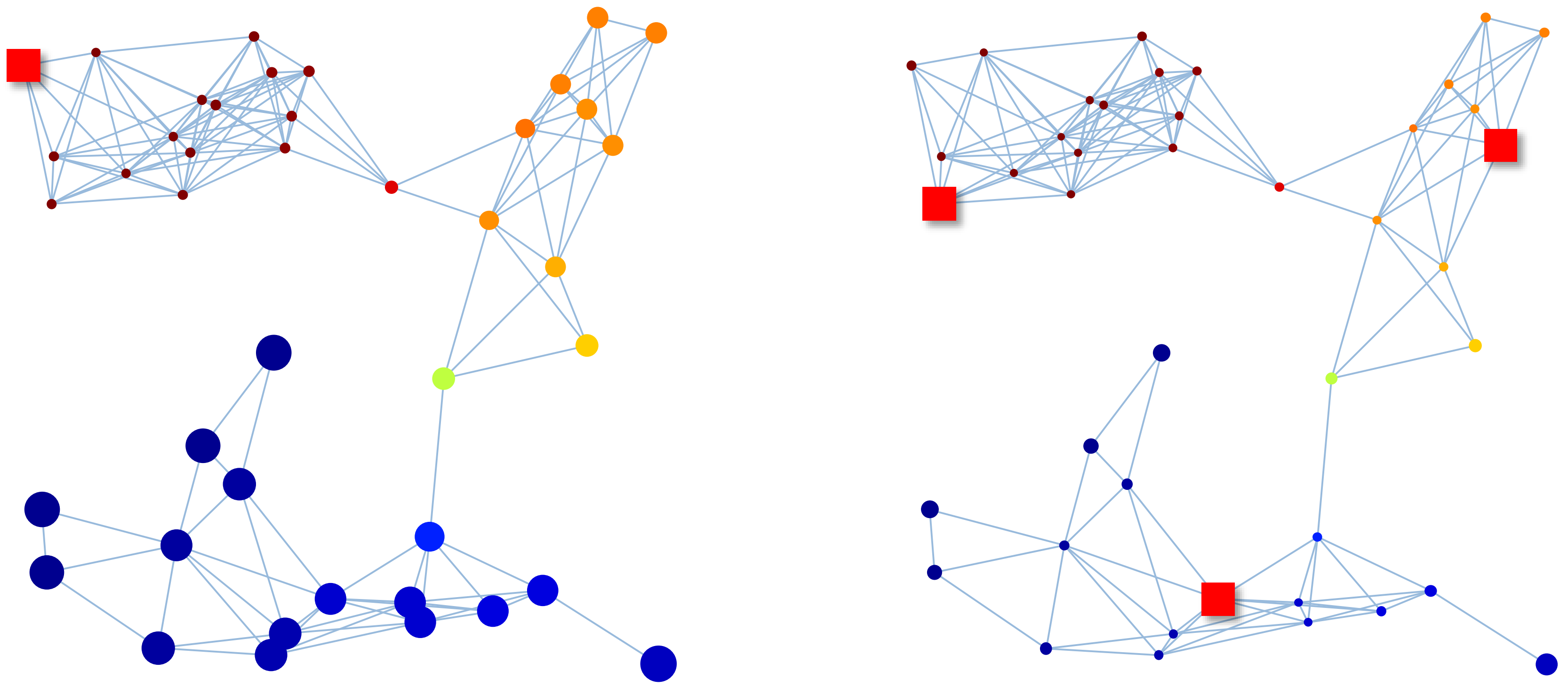
Fundamental limits?



The measurement graph is key

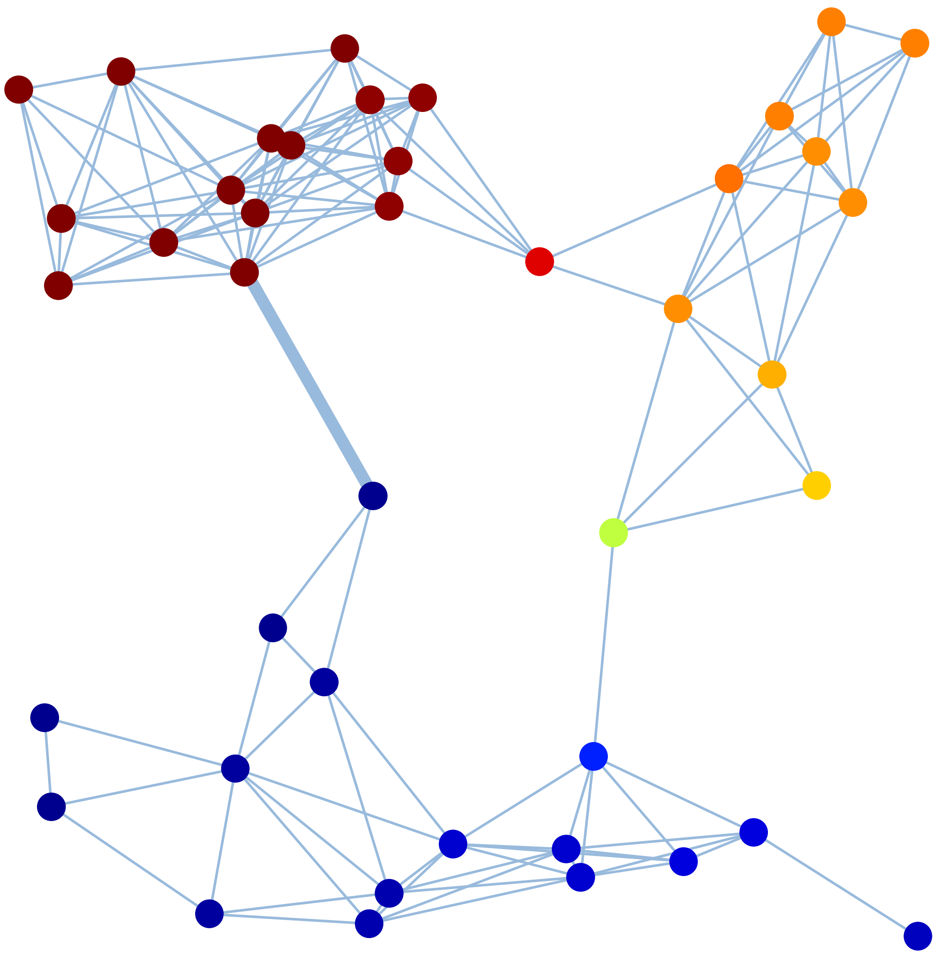


Where should we place anchors?



*Node **size** is proportional to lower-bound on **variance***

How much are new data worth?



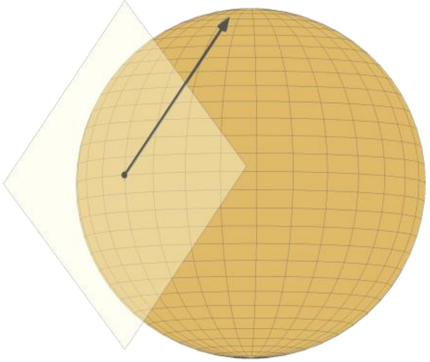
Cramér-Rao bounds (CRB), 1945

Bound on **covariance** C of unbiased estimators:

$$C \succcurlyeq F^{-1}$$

Fisher information F measures the average info in the measurements w.r.t. the parameters.

CRB's on manifolds



The parameter space is a manifold: $\mathrm{SO}(n)^N$

Work on **tangent space**, handle **curvature**
Smith '05

Without anchors, Fisher information is **singular**

Work on the **quotient manifold**
Xavier & Barroso '05

CRB's on manifolds

Approximate CRB:

$$C \succcurlyeq F^{+}$$

Curvature is negligible at reasonable SNR

Pseudo-inversion rightly ignores the **singularity**

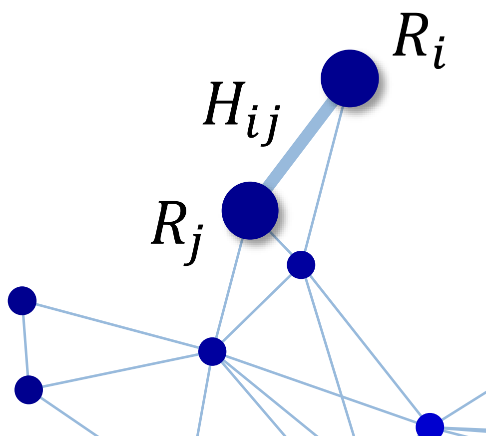
A wide family of noise models

$$H_{ij} = Z_{ij} \cdot R_i R_j^{-1} \quad \text{with} \quad Z_{ij} \sim f_{ij}\colon \mathrm{SO}(n) \to \mathbb{R}$$

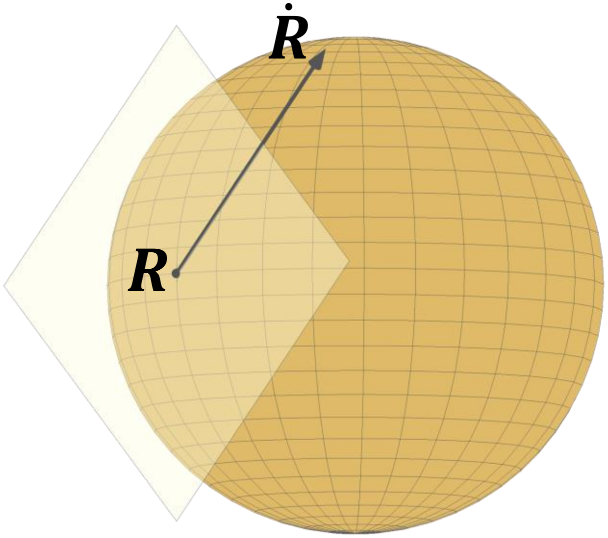
A1. f_{ij} is smooth and positive

A2. Noise is **independent** across edges

A3. $f_{ij}(QZQ^{-1}) = f_{ij}(Z)$
Think: **isotropic** and **unbiased**



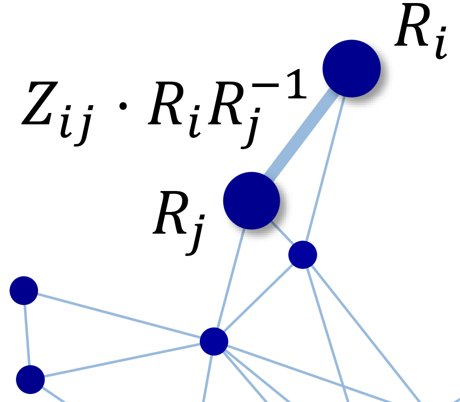
Fisher information measures likelihood sensitivity



$$F\left[\dot{\boldsymbol{R}}, \dot{\boldsymbol{R}}\right] = \mathbf{E}\left\{\mathrm{D}\mathcal{L}(\boldsymbol{R})[\dot{\boldsymbol{R}}]^2\right\}$$

With $\mathcal{L}$ the log-likelihood, assuming **independence**:

$$\mathcal{L}(\widehat{\boldsymbol{R}}) = \sum_{i\sim j} \log f_{ij}(H_{ij}\,\hat{R}_j\hat{R}_i^{-1})$$



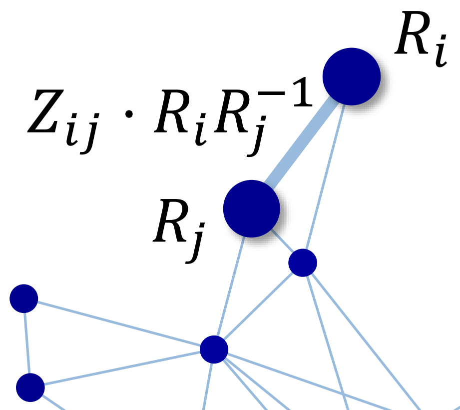
Independence structures F

Non-commutativity makes it a bit technical.

For $\mathrm{SO}(2)$ and Langevin noise: $(f \propto \exp(\kappa \cdot \mathrm{Tr}(Z)))$

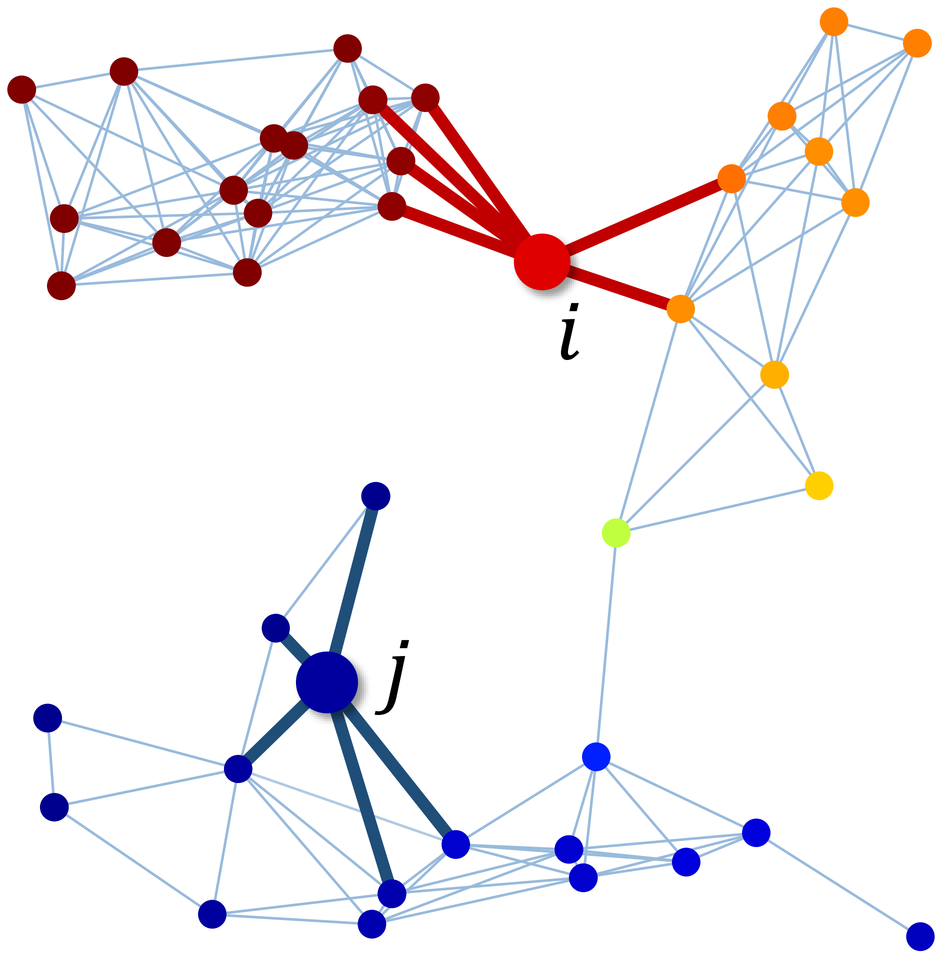
$$F_{ij} = \kappa^2 \sum_{k\sim i} \sum_{\ell \sim j} \mathbf{E}\{\langle Z_{ik}, \Omega \rangle \cdot \langle Z_{j\ell}, \Omega \rangle\}$$

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



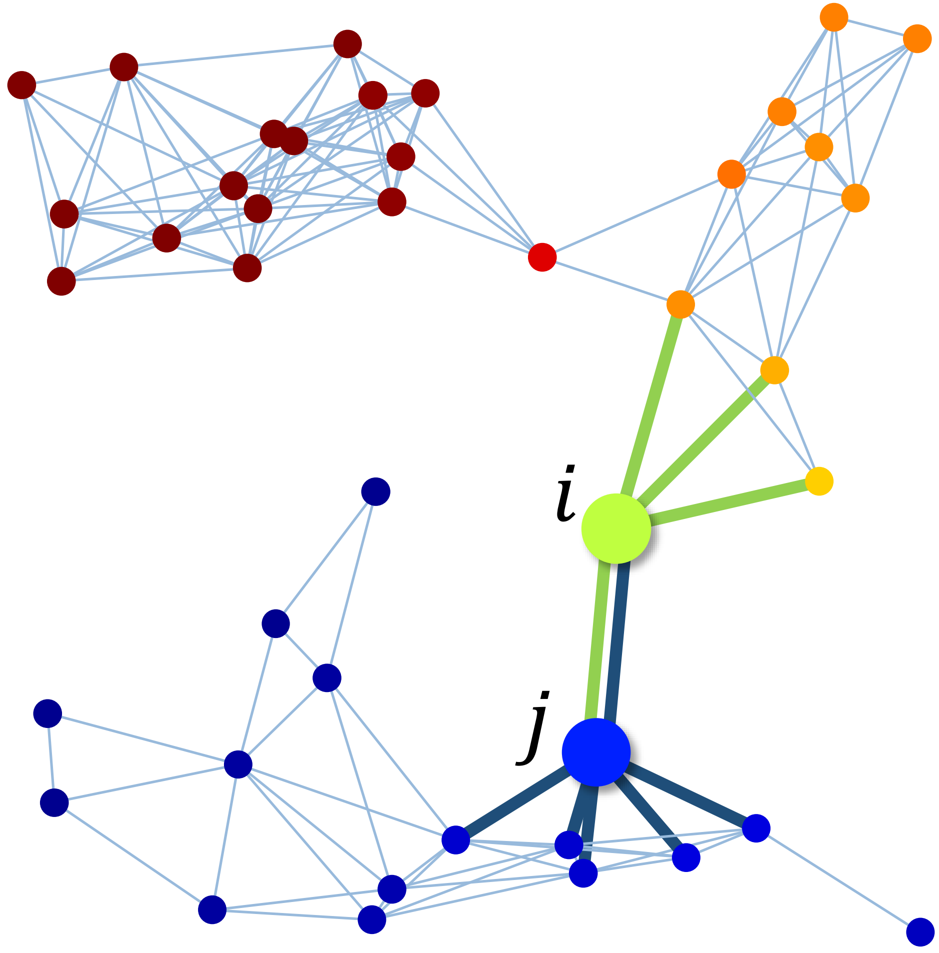
Disjoint nodes contribute a 0

$F_{ij} = 0$



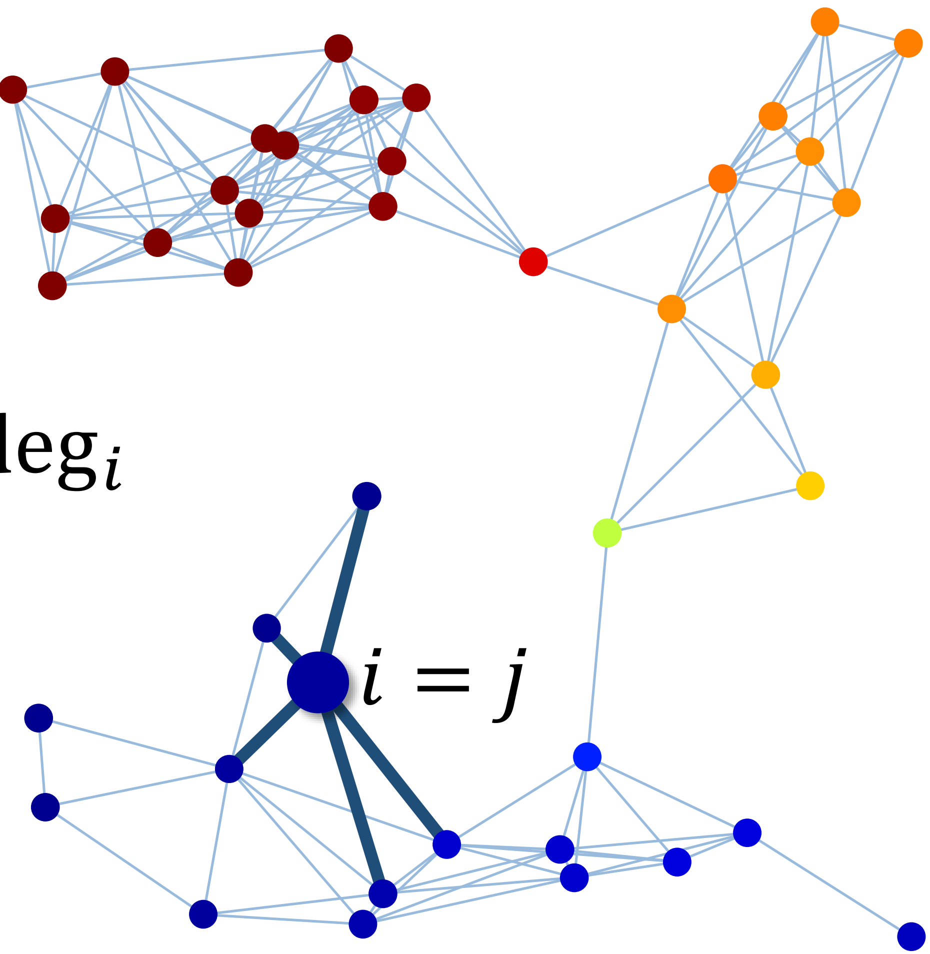
Neighbors contribute $-w$

$$F_{ij} = -w$$



Individuals contribute their degree

$$F_{ii} = w \cdot \text{deg}_i$$



Fisher information is a Laplacian

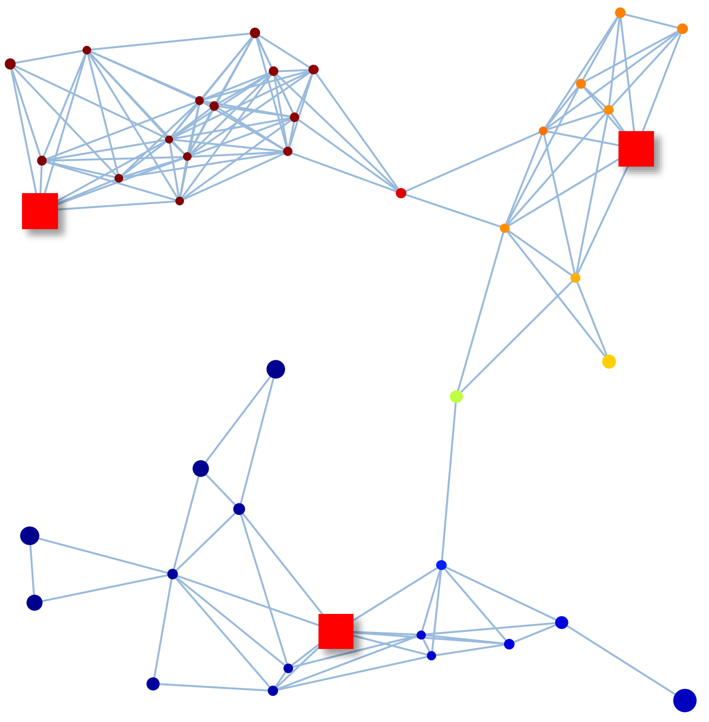
Assuming **independent**, **isotropic** and **unbiased** noise,

$$F = L = D - A$$

is the **weighted Laplacian** of the graph.

Remarkably, F does not depend on the true rotations.

The anchored case



$$C \succcurlyeq L^{+}$$

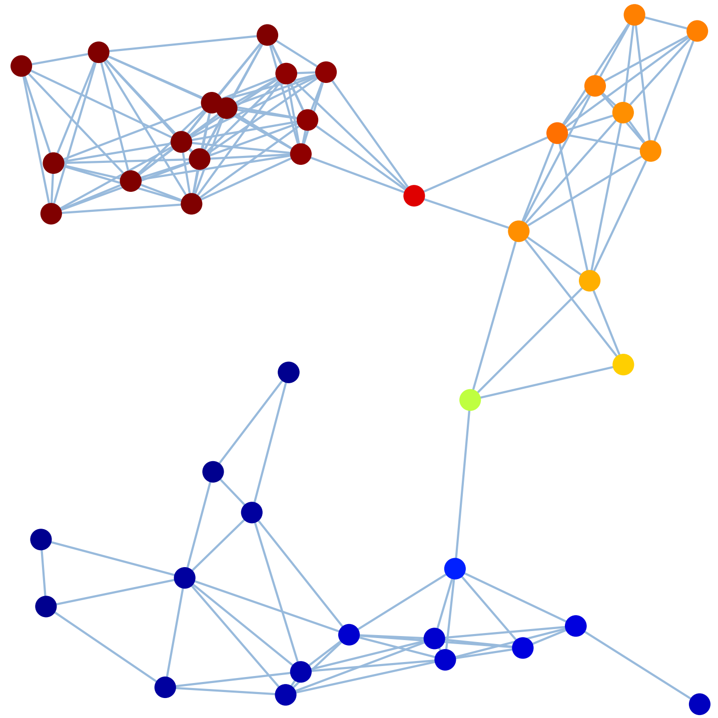
For individual nodes:

$$\mathbf{E}\{\mathrm{dist}^2(R_i, \hat{R}_i)\} \geq d \cdot L^{+}_{ii}$$

Random walk from *i* to any anchor.

*Node **size** is proportional to lower-bound on **variance***

The anchor-free case



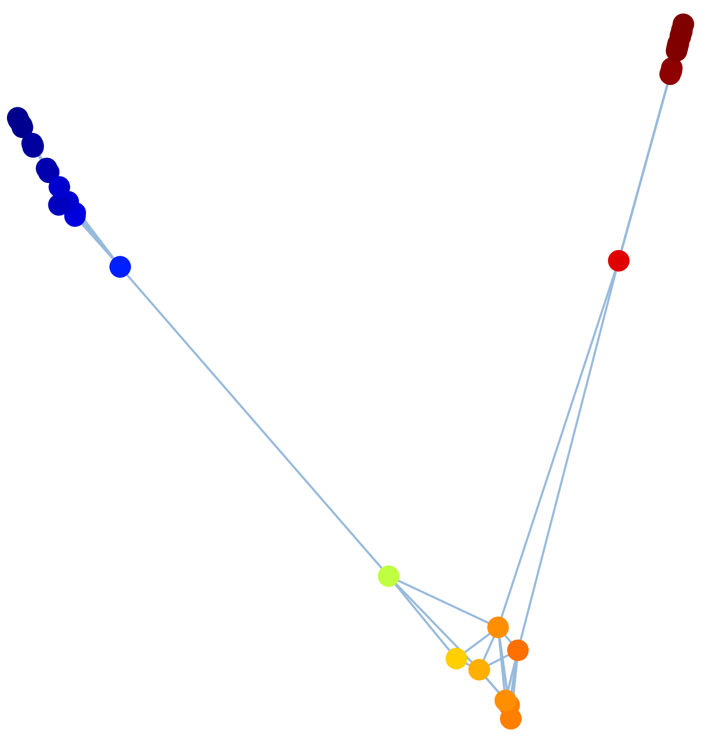
$$C \succcurlyeq L^{+}$$

For pairs of nodes:

$$\mathbf{E}\left\{\operatorname{dist}^2(R_i R_j^{-1}, \hat{R}_i \hat{R}_j^{-1})\right\} \geq d \cdot \left(e_i - e_j\right)^T L^{+} \left(e_i - e_j\right)$$

Random walk from *i* to *j* and back again.

The anchor-free case



$$C \succcurlyeq L^{+}$$

For pairs of nodes:

$$\mathbf{E}\left\{\mathrm{dist}^2\left(R_i R_j^{-1}, \hat{R}_i \hat{R}_j^{-1}\right)\right\}$$
$$\geq d \cdot \left(e_i - e_j\right)^T L^{+}\left(e_i - e_j\right)$$

Random walk from *i* to *j* and back again.

Conclusions

The Laplacian-structured CRB gives a firm **qualitative** and **quantitative** grasp on synchronisation.

This was understood for the translation group and for in-plane rotations: both commutative.

This work moves to a non-commutative group.