Low-rank approaches for SDP's in community detection

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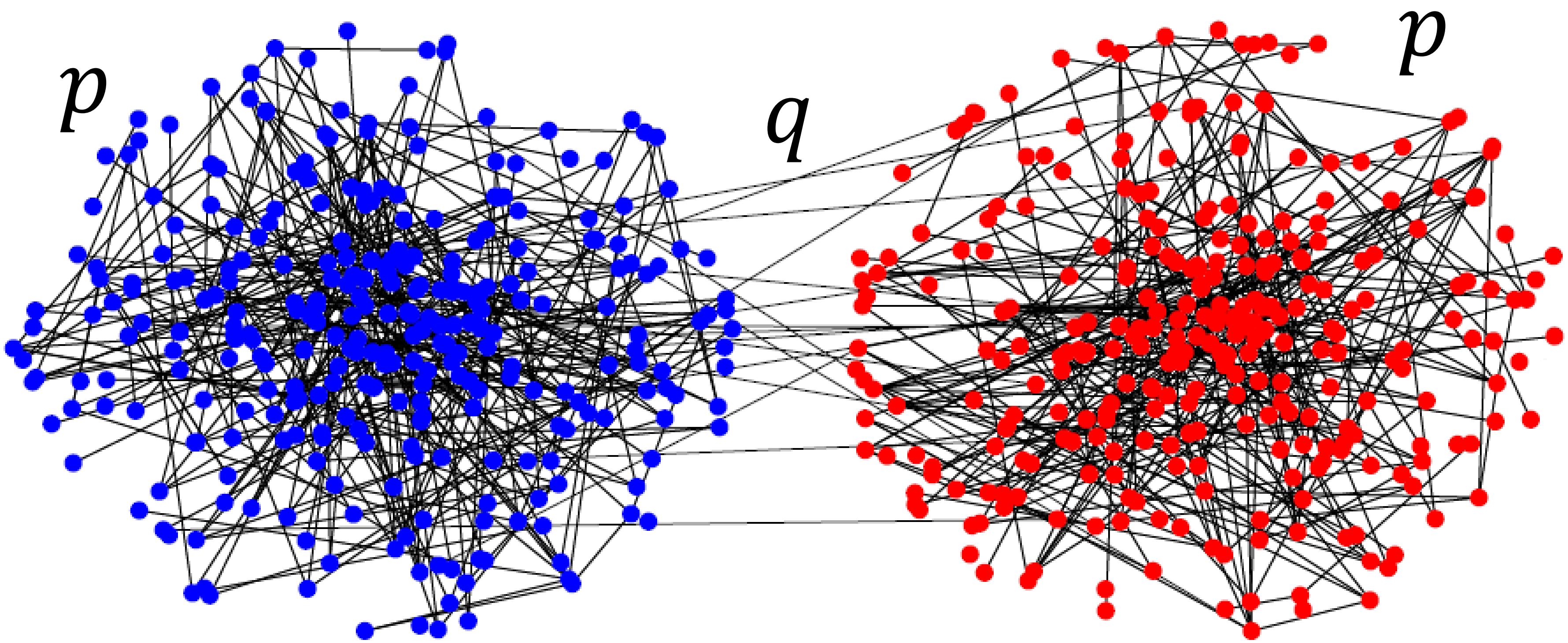
Community detection in the stochastic block model (SBM)



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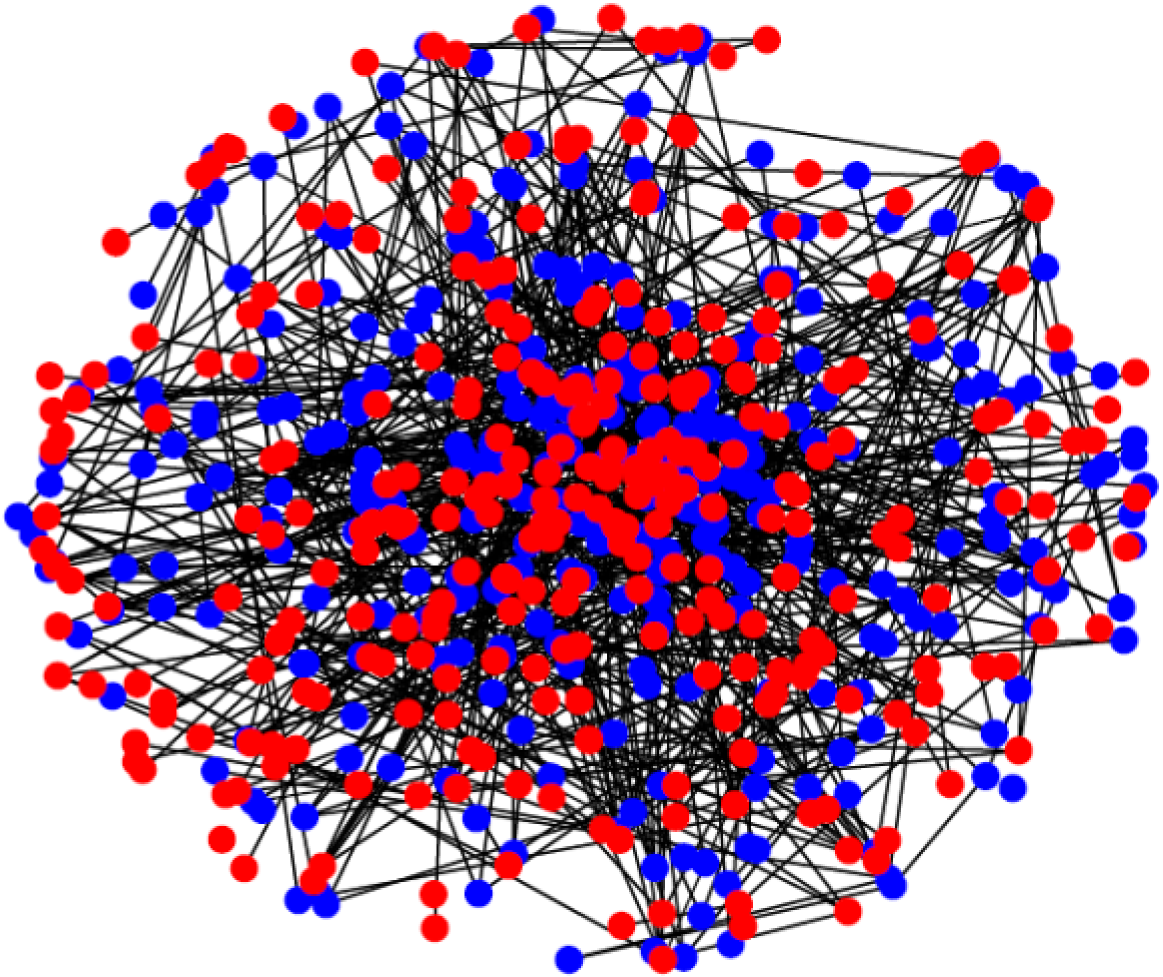

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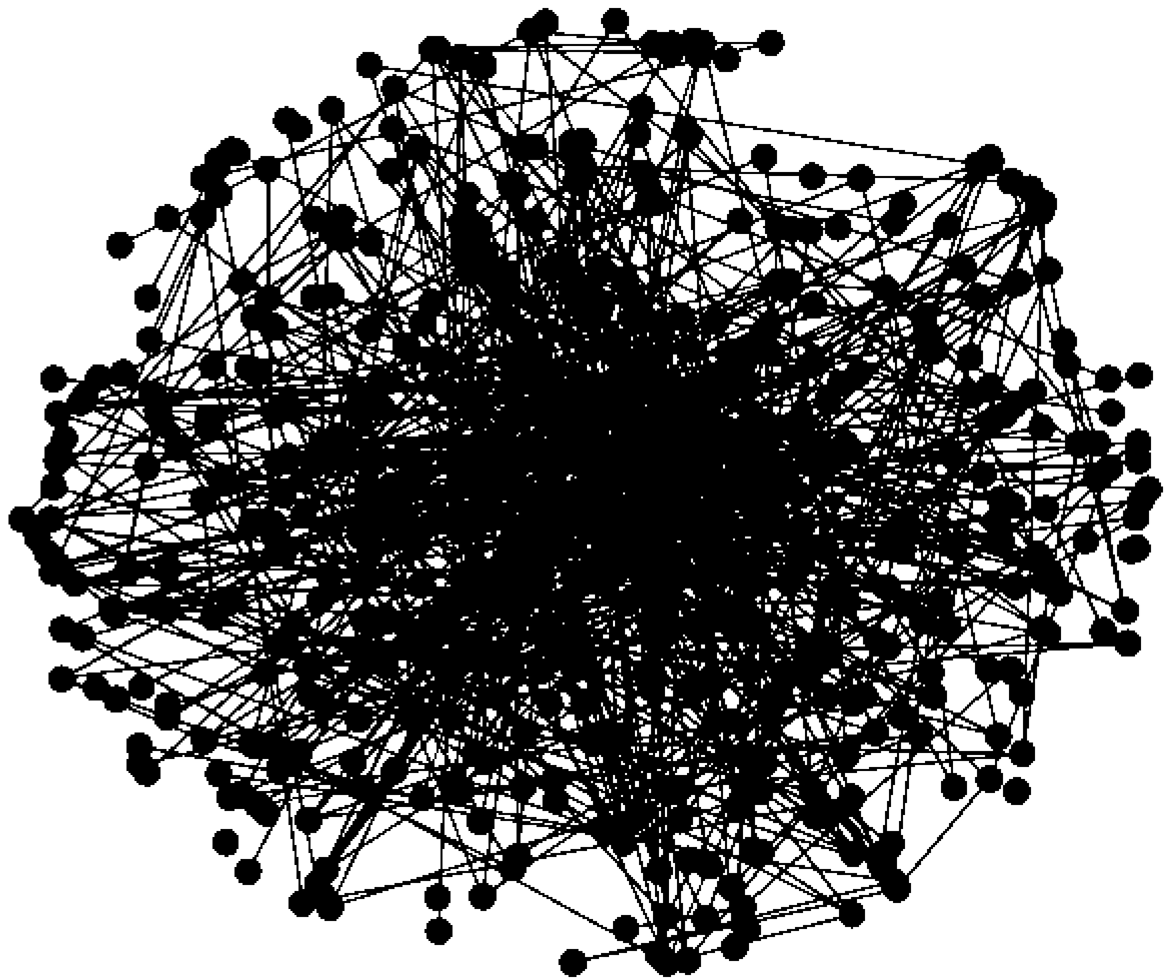


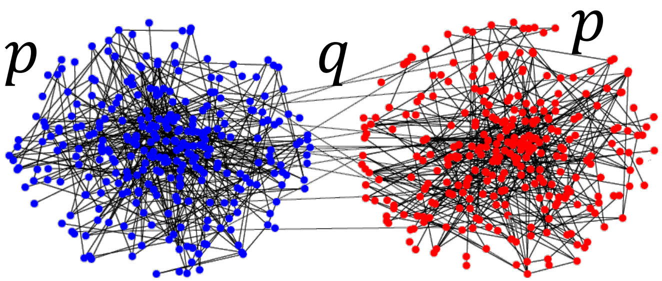
The constant average degree regime

Link within: $p = \frac{a}{n}$

Link across: $q = \frac{b}{n}$

Each community has a giant connected component, whp.

Hope for non-trivial correlation with true partition.



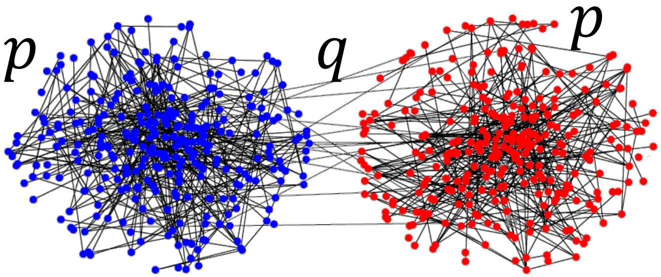
The dense regime

Link within: $p = \alpha \frac{\log n}{n}$

Link across: $q = \beta \frac{\log n}{n}$

Each community is connected, whp.

Hope for exact recovery.



The key SNR quantity: $\lambda(p,q)$

$$\lambda(p,q) = \frac{p-q}{\sqrt{2(p+q)}}\sqrt{n}$$

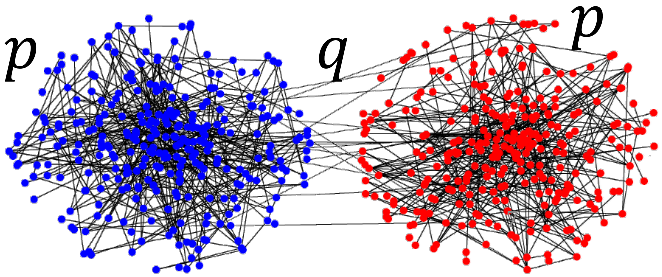
For non-trivial correlation: need $\lambda > 1$

Decelle et al. ‘11, Mossel et al. ‘14, Massoulié et al. ‘14

For exact recovery: “need $\lambda > \sqrt{2\log n}$”

Mossel et al. ‘14, Abbe et al. ’14

(Precise condition: $\sqrt{\alpha} - \sqrt{\beta} > \sqrt{2}$.)



Relaxation of MLE gives SDP for SBM

With A the adjacency matrix and $A' = A - \frac{p+q}{2}\mathbf{1}\mathbf{1}^T$:

$$\max_X \langle A', X\rangle \ \text{s.t.}\ \operatorname{diag}(X) = \mathbf{1}, X \succcurlyeq 0$$

Non-trivial correlation:

$\forall\delta > 0$, if $\frac{p+q}{2}n$ large enough, $\lambda > 1 + \delta$ is enough

Guedon & Vershynin '14, Montanari & Sen '15, Javanmard et al. '15

Exact recovery: SDP is tight at the info limit

Hajek et al. '14, Bandeira '15

The Burer–Monteiro approach

$$\max_X \langle A', X \rangle \text{ s.t. } \operatorname{diag}(X) = \mathbf{1}, X \succcurlyeq 0$$

Parameterize $X = YY^T$ with Y of size $n \times p$:

$$\max_Y \left\langle A', YY^T \right\rangle \text{ s.t. } \operatorname{diag}\left(YY^T\right) = \mathbf{1}$$

The aggressive version: $p = 2$.

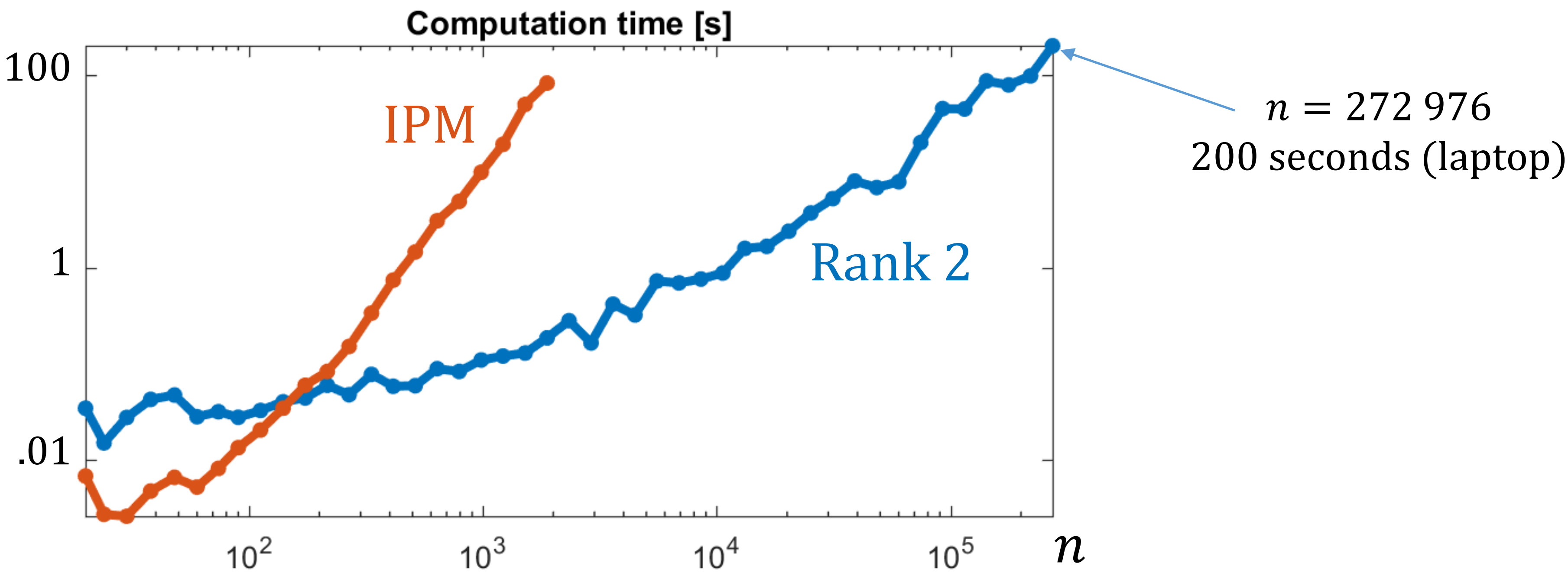
Non-convex optimization on the n-torus

$$\max_{Y \in \mathbf{R}^{n\times 2}} \langle A', YY^T \rangle \text{ s.t. } \operatorname{diag}(YY^T) = \mathbf{1}$$

Low-dimensional, and no conic constraint.

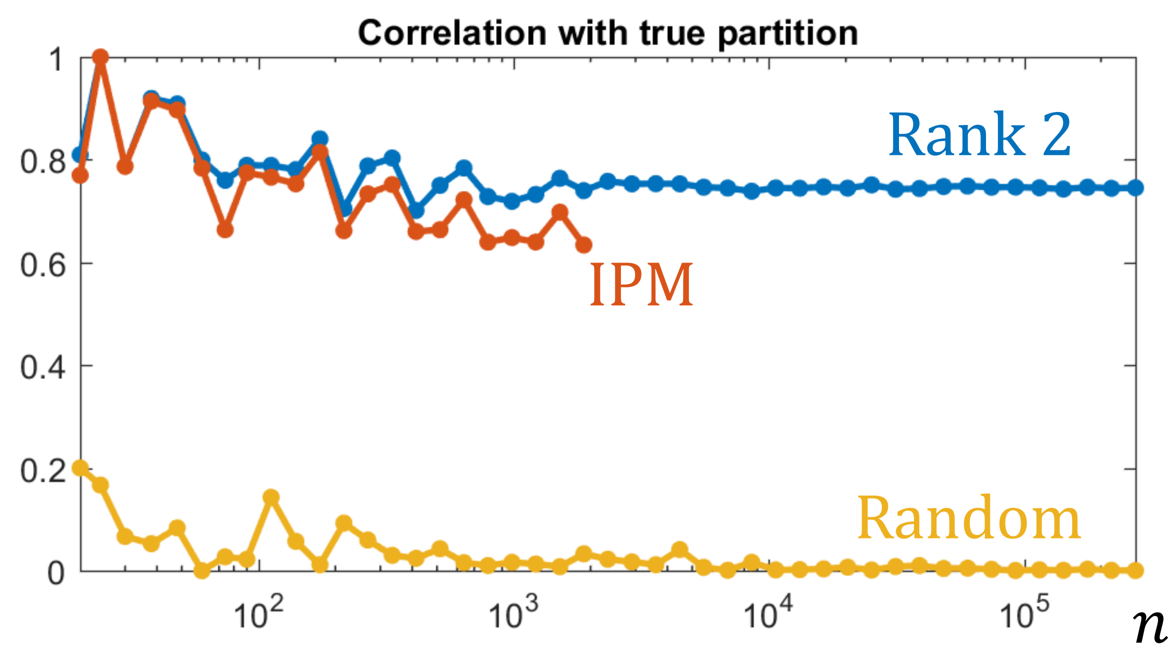
We run Riemannian trust regions via Manopt.

Do KKT points have good statistical properties?



$$p = \frac{10}{n}, q = \frac{2}{n}$$

$$\lambda = 1.63$$



Main result 1: non-trivial correlation

$$\max_{Y \in \mathbf{R}^{n\times 2}} \langle A', YY^T \rangle \text{ s.t. } \operatorname{diag}(YY^T) = \mathbf{1}$$

In the constant average degree regime, for any $\delta > 0$, if $\frac{p+q}{2}n$ is large enough and $\lambda > 8 + \delta$,

Then, there exists $\varepsilon > 0$ such that, whp, all second order KKT points Y correlate non-trivially with the true partition $\boldsymbol{g}$:

$$\frac{1}{n}\left\|Y^T \boldsymbol{g}\right\|_2 \geq \varepsilon.$$

Main result 2: exact recovery

$$\max_{Y \in \mathbf{R}^{n \times 2}} \langle A', YY^T \rangle \text{ s.t. } \operatorname{diag}(YY^T) = \mathbf{1}$$

There exists c (universal) such that, if

$$\lambda \geq cn^{1/3},$$

then, whp, all second-order KKT points Y are optimal and correspond to $\boldsymbol{g}$:

$$YY^T = \boldsymbol{g}\boldsymbol{g}^T.$$

1. $\operatorname{Hess} f(Y) \succcurlyeq 0 \Leftrightarrow \operatorname{ddiag}(A'YY^T) \succcurlyeq A' \circ YY^T$

$$\langle \operatorname{ddiag}(A'YY^T), \boldsymbol{g}\boldsymbol{g}^T \circ YY^T \rangle \geq \langle A' \circ YY^T, \boldsymbol{g}\boldsymbol{g}^T \circ YY^T \rangle$$

2. Link A' to the signal: $A' \propto \boldsymbol{g}\boldsymbol{g}^T + \frac{n}{\lambda} E + D$

$$\left\langle \boldsymbol{g}\boldsymbol{g}^T + \frac{n}{\lambda} E, YY^T \right\rangle \geq \left\langle \boldsymbol{g}\boldsymbol{g}^T + \frac{n}{\lambda} E, \boldsymbol{g}\boldsymbol{g}^T \circ YY^T \circ YY^T \right\rangle$$

3. Use noise property: $\max_{X \succcurlyeq 0, \operatorname{diag}(X) = \mathbf{1}} \langle E, X \rangle \leq (2 + o_d(1))n$

$$\|Y^T \boldsymbol{g}\|^2 \geq \|YY^T\|_F^2 - \frac{2n^2}{\lambda}(2 + o_d(1)) \geq n^2 \left(\frac{1}{2} - \frac{4 + o_d(1)}{\lambda} \right)$$

Note: Sufficient to ensure $\operatorname{Hess} f(Y) \succcurlyeq -\varepsilon \cdot \mathrm{I}$

More of this in statistics?

It’s a tempting family of estimators.

Toward computation bounds: see global rates of convergence to KKT points on manifolds: 1605.08101

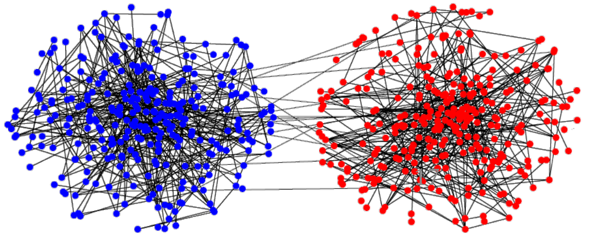
More guarantees for Burer–Monteiro approach to ‘smooth’ SDP’s: 1606.04970

Maximum likelihood estimation for SBM is combinatorial optimization

$A \in \mathbf{R}^{n\times n}$ is the observed adjacency matrix:

$$\max_{\boldsymbol{z}} \langle A, \boldsymbol{z}\boldsymbol{z}^T \rangle \quad \text{s.t.} \quad \boldsymbol{z} \in \{\pm 1\}^n \ \text{ and } \ \mathbf{1}^T\boldsymbol{z} = 0$$

Hard problem for general A: need to relax.



Step 1: remove the linear constraint

$$\max_{\boldsymbol{z}} \langle A, \boldsymbol{z}\boldsymbol{z}^T \rangle \quad \text{s.t.} \quad \boldsymbol{z} \in \{\pm 1\}^n \ \text{ and } \ \mathbf{1}^T\boldsymbol{z} = 0$$

If $\boldsymbol{g} \in \{\pm 1\}^n$ is the true partition, then

$$\mathbf{E}\{A\} = \frac{p+q}{2}\mathbf{1}\mathbf{1}^T + \frac{p-q}{2}\boldsymbol{g}\boldsymbol{g}^T.$$

Remove the bias toward $\mathbf{1}$: $A' = A - \frac{p+q}{2}\mathbf{1}\mathbf{1}^T$

Step 2: relax to a semidefinite program

$$\max_{\boldsymbol{z}} \left\langle A', \boldsymbol{z}\boldsymbol{z}^T \right\rangle \ \text{ s.t. } \ \boldsymbol{z} \in \{\pm 1\}^n$$

Binary constraints $\boldsymbol{z} \in \{\pm 1\}^n$
are equivalent to $\text{diag}\left(\boldsymbol{z}\boldsymbol{z}^T\right) = \mathbf{1}$.

Introduce $X = \boldsymbol{z}\boldsymbol{z}^T$. Equivalent problem:

$$\max_X \langle A', X \rangle \ \text{ s.t. } \text{diag}(X) = \mathbf{1}, X \succcurlyeq 0, \text{rank}(X) = 1$$