Discrete curve fitting on manifolds

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Université catholique de Louvain

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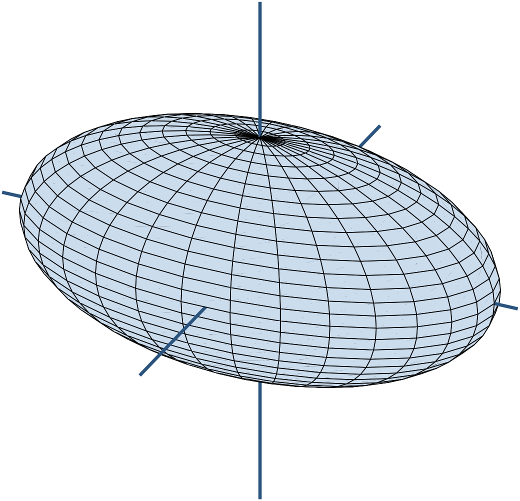
Motivation: interpolation on $\mathrm{SO}(3)$



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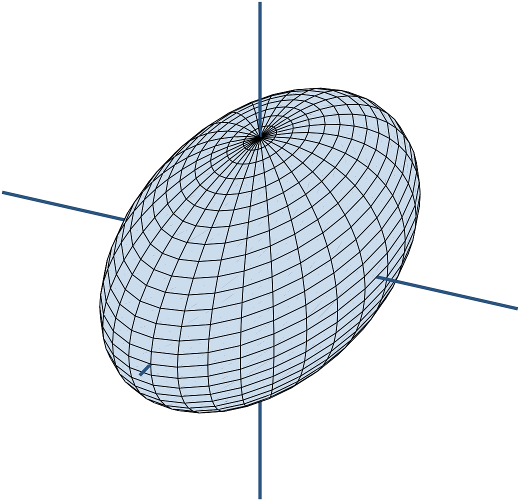

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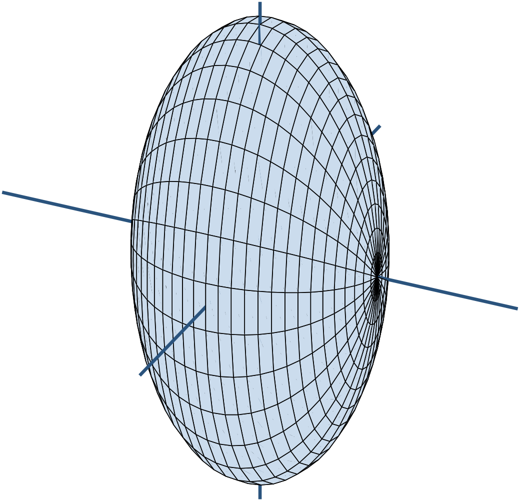

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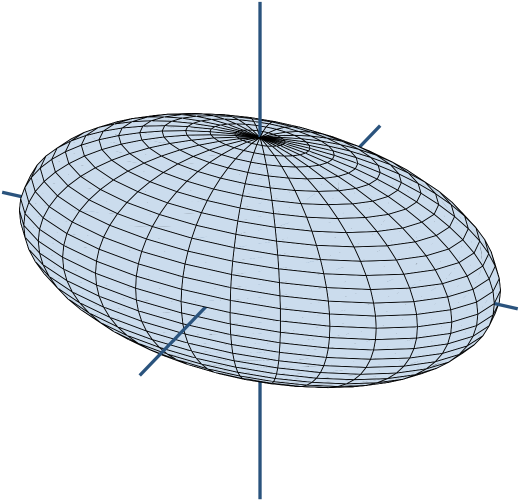

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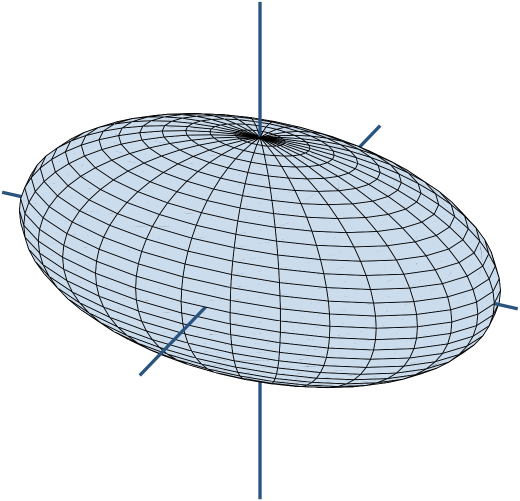

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A balance between fitting and smoothness

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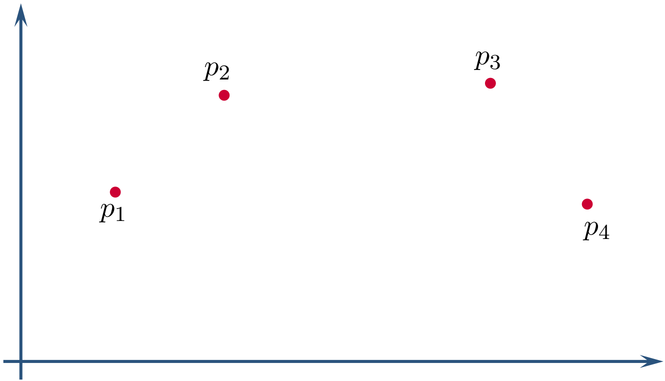

Each data point p_i corresponds to a fixed time t_i.

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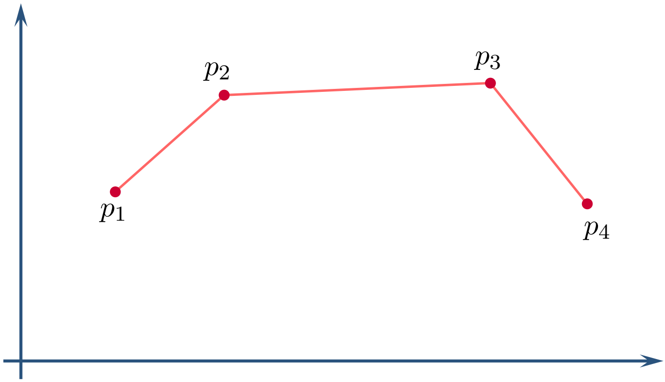

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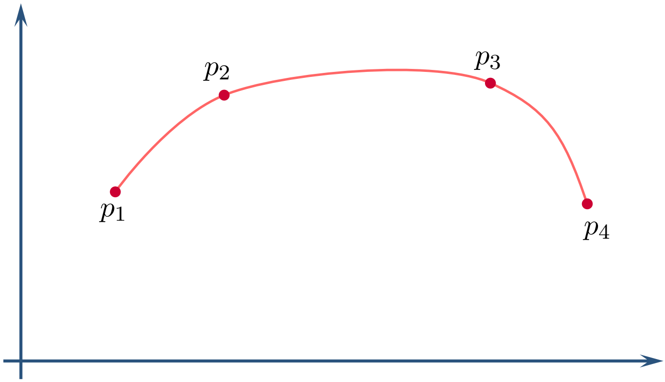

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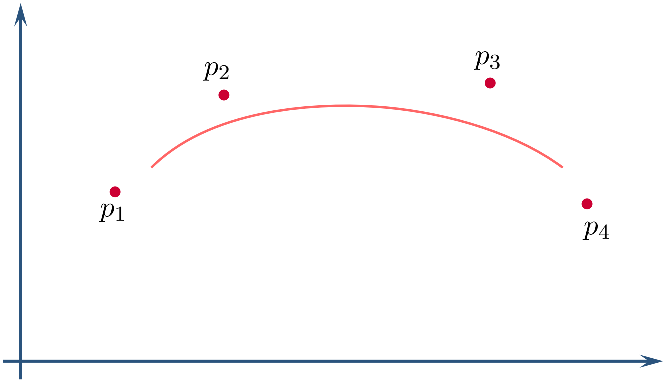

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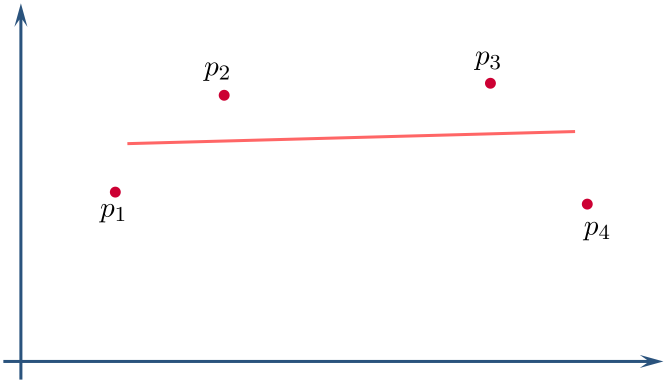


Regression is about denoising and filling the gaps.

The regression problem in $\mathbb{R}^2$

can be seen as an optimization problem

Minimize:

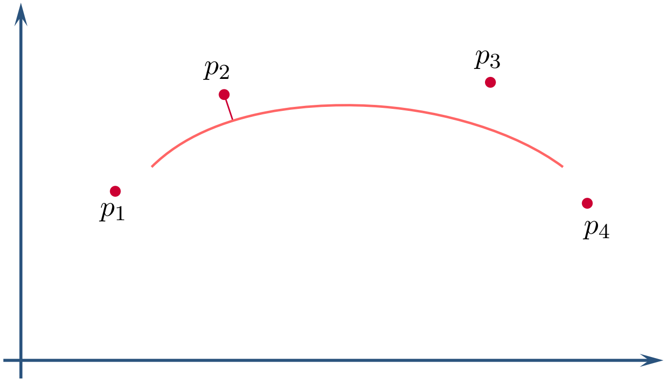
$$\hat{E}(\gamma) = \underbrace{\sum_{i=1}^{N} \|p_i - \gamma(t_i)\|^2}_{\text{Penalty on misfit}} + \lambda \underbrace{\int_{t_1}^{t_N} \|\dot{\gamma}(t)\|^2 \mathrm{d}t}_{\text{Penalty on velocity}} + \mu \underbrace{\int_{t_1}^{t_N} \|\ddot{\gamma}(t)\|^2 \mathrm{d}t}_{\text{Penalty on acceleration}}$$

λ and μ (≥ 0) balance fitting VS smoothness.

Minimize over some curve space $\hat{\Gamma}$: $\dim \hat{\Gamma}$ may be infinite.

We discretize the curves γ

hence reverting to finite dimensional optimization



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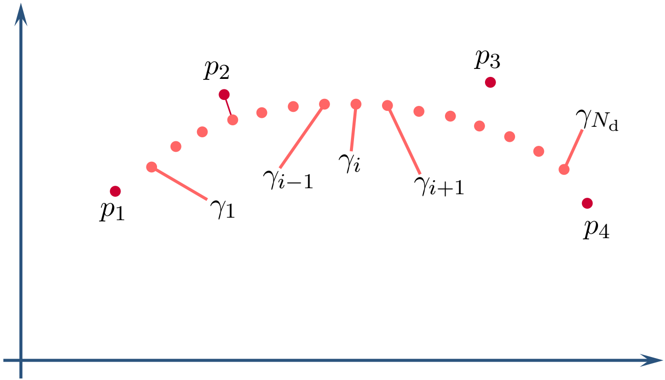


Each point γ_i corresponds to a fixed time τ_i

$\Gamma = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \equiv \mathbb{R}^{N_\mathrm{d} \times n}$

We thus need a new objective E

defined over the new curve space Γ

$$\sum_{i=1}^{N} \|p_i - \gamma(t_i)\|^2 \qquad \int_{t_1}^{t_N} \|\dot{\gamma}(t)\|^2 \mathrm{d}t \qquad \int_{t_1}^{t_N} \|\ddot{\gamma}(t)\|^2 \mathrm{d}t$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$E(\gamma) = \sum_{i=1}^{N} \|p_i - \gamma_{s_i}\|^2 + \lambda \sum_{i=1}^{N_\mathrm{d}} \alpha_i \|v_i\|^2 + \mu \sum_{i=1}^{N_\mathrm{d}} \beta_i \|a_i\|^2$$

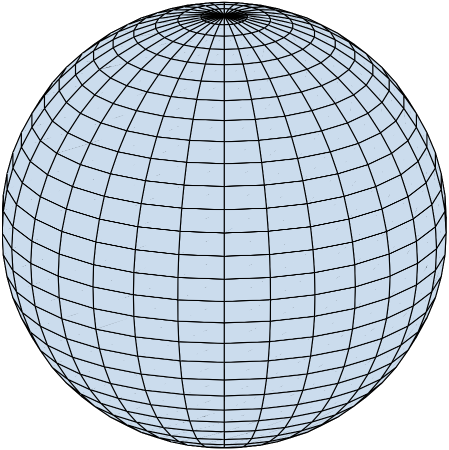
What if the data lies on a manifold?

Manifolds are smoothly "curved" spaces.

Simple toy example: the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$

More exciting manifolds discussed in this work: $\mathbb{P}_+^n$ and $\mathrm{SO}(n)$.

The regression problem on $\mathbb{S}^2$



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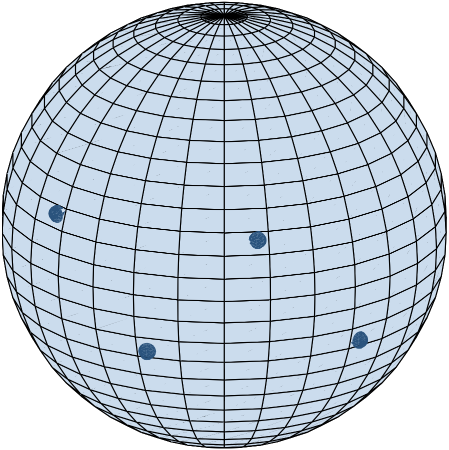

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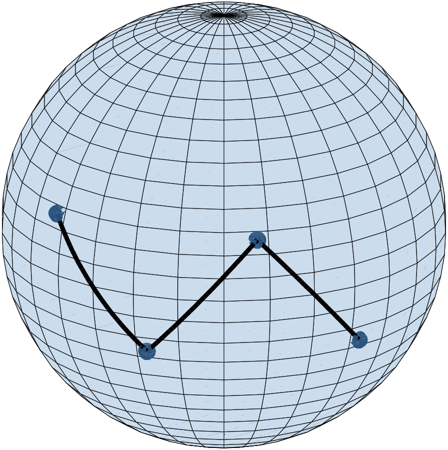

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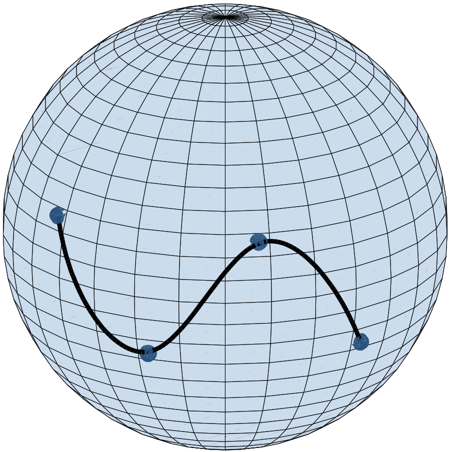

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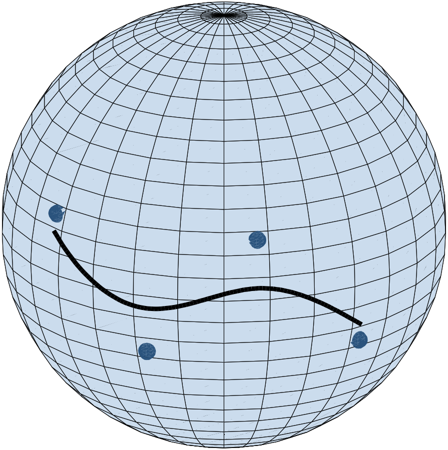

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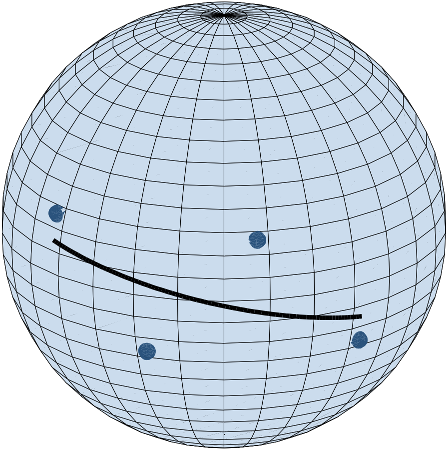

We need a few concepts from Riemannian geometry

to define discrete regression on $\mathbb{S}^2$

Redefine E over $\Gamma = \mathbb{S}^2 \times \cdots \times \mathbb{S}^2$:

$$E(\gamma) = \boxed{\sum_{i=1}^{N} \|p_i - \gamma_{s_i}\|^2} + \lambda \sum_{i=1}^{N_\mathrm{d}} \alpha_i \|v_i\|^2 + \mu \sum_{i=1}^{N_\mathrm{d}} \beta_i \|a_i\|^2$$

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but $\mathbb{S}^2$ is not a vector space :(

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Now, we can interpret the terms:

$\gamma_{i+1} - \gamma_i$ *is a vector rooted at* γ_i *and pointing toward* γ_{i+1}

Logarithms on manifolds generalize differences

We use them to define geometric finite differences

$\mathrm{Log}_a(b)$ *is a vector rooted at* a*, in the tangent space to* $\mathbb{S}^2$ *at* a*, pointing toward* b*. Furthermore,* $\|\mathrm{Log}_a(b)\| = \mathrm{dist}(a,b)$*.*

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$$b - a \quad \text{is replaced by} \quad \operatorname{Log}_a(b)$$

Hence:

$$v_i = \frac{\operatorname{Log}_{\gamma_i}(\gamma_{i+1})}{\Delta\tau} \qquad\qquad a_i = \frac{\operatorname{Log}_{\gamma_i}(\gamma_{i+1}) + \operatorname{Log}_{\gamma_i}(\gamma_{i-1})}{\Delta\tau^2}$$

We now have a proper objective for manifolds

$$E(\gamma) = \overset{\text{Penalty on misfit}}{\boxed{\sum_{i=1}^{N} \operatorname{dist}^2(p_i, \gamma_{s_i})}} + \lambda \overset{\text{Penalty on velocity}}{\boxed{\sum_{i=1}^{N_\mathrm{d}-1} \alpha_i \left\| \frac{\operatorname{Log}_{\gamma_i}(\gamma_{i+1})}{\Delta\tau} \right\|^2}}$$
$$+ \mu \underset{\text{Penalty on acceleration}}{\boxed{\sum_{i=2}^{N_\mathrm{d}-1} \beta_i \left\| \frac{\operatorname{Log}_{\gamma_i}(\gamma_{i+1}) + \operatorname{Log}_{\gamma_i}(\gamma_{i-1})}{\Delta\tau^2} \right\|^2}}$$

Minimize over $\Gamma = \mathbb{S}^2 \times \cdots \times \mathbb{S}^2$, a finite dimensional manifold.

The constraint $\gamma \in \Gamma$ is tough for standard software.

To minimize E, we exploit the geometry

by generalizing unconstrained descent methods

Geometric steepest descent step:

1. Compute the steepest descent direction;
2. Choose a step length along the corresponding geodesic;
3. Make the step while remaining on the manifold.

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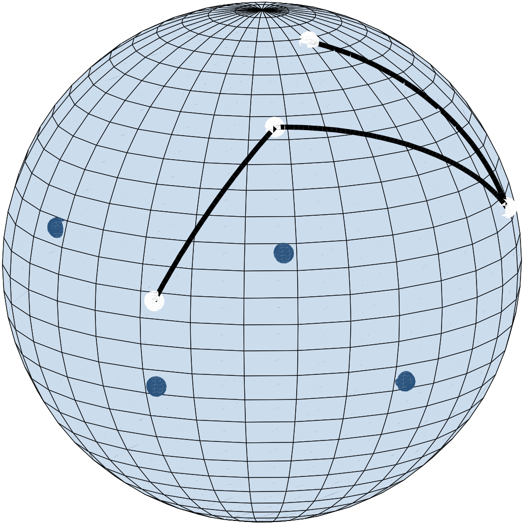
2. Then generalized it to manifolds as an optimization problem on Γ;
3. And described an optimization scheme on manifolds.

Example of convergence on $\mathbb{S}^2$

with geometric non-linear CG and iterative refinement



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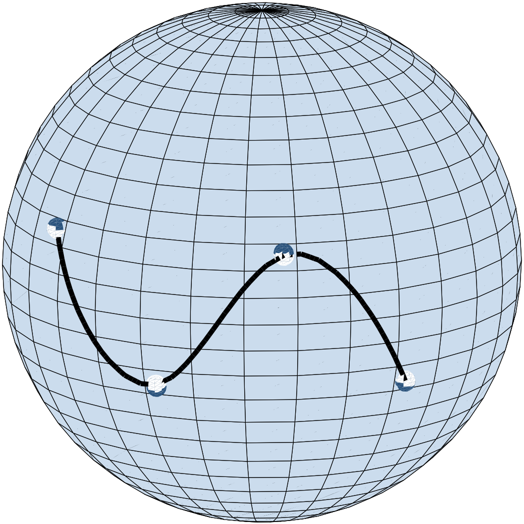

What's so hard about it?

The main constraint: tractability of the manifold.

It can be slow. . .

We are currently trying second order methods.

. . . or even impossible.

Our method is well-defined on any Riemannian manifold, but is only practical on the gentle ones.

What's next?

Applications?