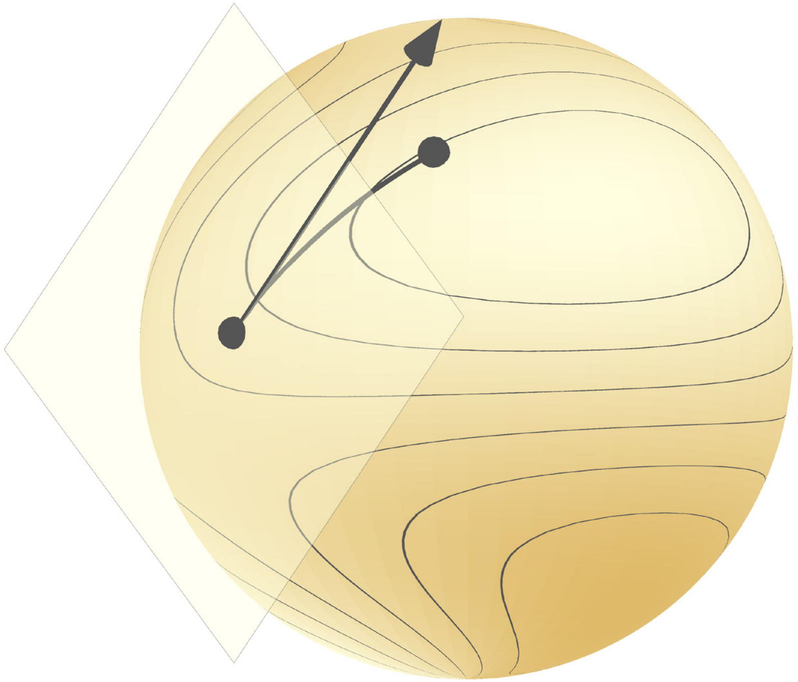
Slides and links: nicolasboumal.net/SIAMOP2023

A tutorial on Riemannian optimization

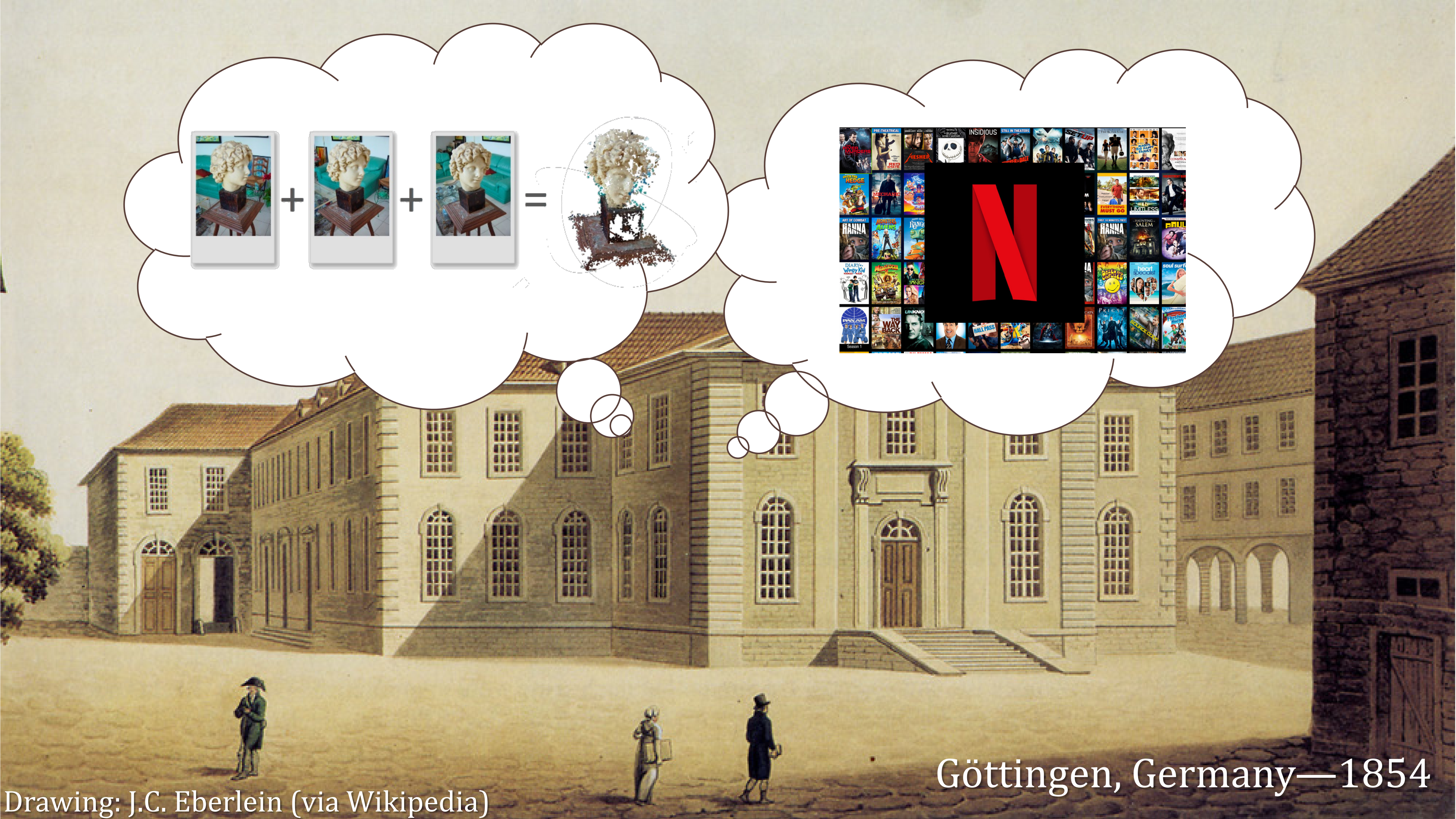
Context, geometry, algorithms, resources



SIAM Conference on Optimization, June 2023

Nicolas Boumal – chair of continuous optimization

Institute of Mathematics, EPFL



Göttingen, Germany—1854

Drawing: J.C. Eberlein (via Wikipedia)

Step 0 in optimization

It starts with a set S and a function $f: S \to \mathbf{R}$. We want to compute:

$$\min_{x \in S} f(x)$$

These bare objects fully specify the problem.

Any additional structure on S and f may (and should) be exploited for algorithmic purposes but is not part of the problem.

Classical unconstrained optimization

The search space *is* a linear space, e.g., $S = \mathbf{R}^n$:

$$\min_{x \in \mathbf{R}^n} f(x)$$

We can *choose* to turn $\mathbf{R}^n$ into a Euclidean space: $\langle u, v \rangle = u^\top v$.

If f is differentiable, we have a gradient $\mathrm{grad} f$ and Hessian $\mathrm{Hess} f$.
We can build algorithms with them: gradient descent, Newton's...

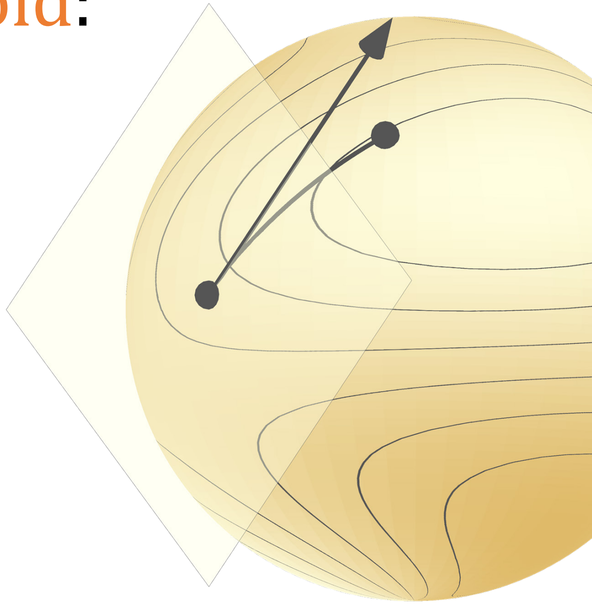
$$\langle \mathrm{grad} f(x), v \rangle = \mathrm{D} f(x)[v] = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

$$\mathrm{Hess} f(x)[v] = \mathrm{D}(\mathrm{grad} f)(x)[v] = \lim_{t \to 0} \frac{\mathrm{grad} f(x + tv) - \mathrm{grad} f(x)}{t}$$

This tutorial: optimization on manifolds

We target applications where $S = \mathcal{M}$ *is* a smooth manifold:

$$\min_{x \in \mathcal{M}} f(x)$$



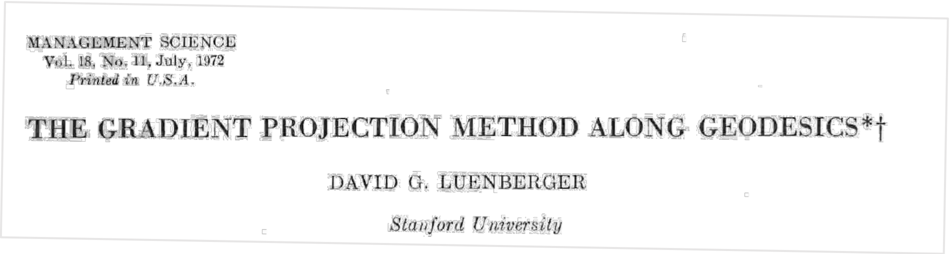
We can *choose* to turn $\mathcal{M}$ into a Riemannian manifold.

If f is differentiable, we have a Riemannian gradient and Hessian.

We can build algorithms with them: gradient descent, Newton's...

Fifty years

Proposed by Luenberger in 1972.


MANAGEMENT SCIENCE
Vol. 18, No. 11, July, 1972
Printed in U.S.A.

THE GRADIENT PROJECTION METHOD ALONG GEODESICS*†

DAVID G. LUENBERGER

Stanford University

Practical since the 1990s
with numerical linear algebra.


SIAM J. MATRIX ANAL. APPL.
Vol. 20, No. 2, pp. 303–353

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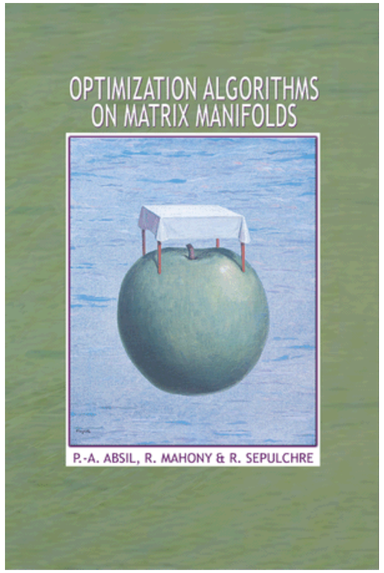
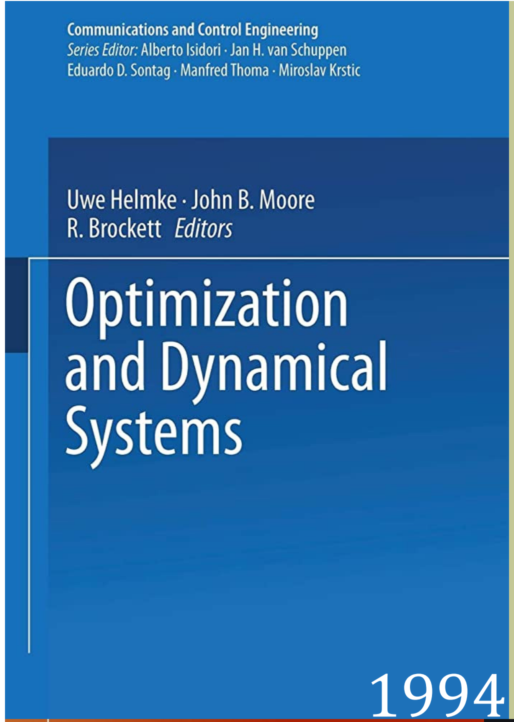
THE GEOMETRY OF ALGORITHMS WITH ORTHOGONALITY CONSTRAINTS*

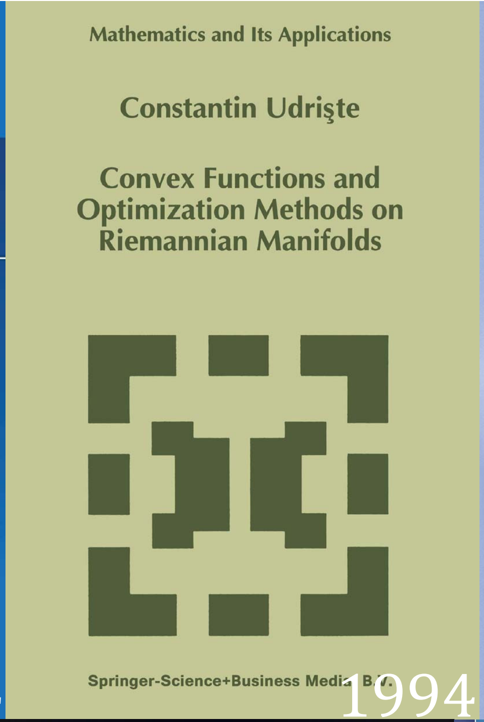
ALAN EDELMAN†, TOMÁS A. ARIAS‡, AND STEVEN T. SMITH§

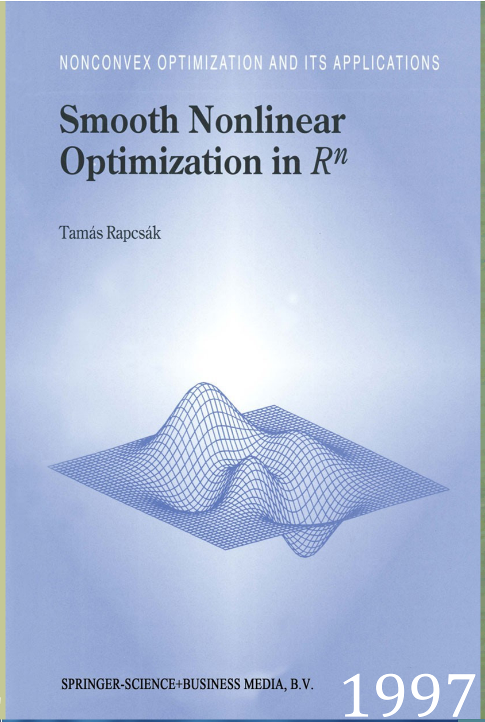
Popularized in the 2010s
by Absil, Mahony & Sepulchre's book.

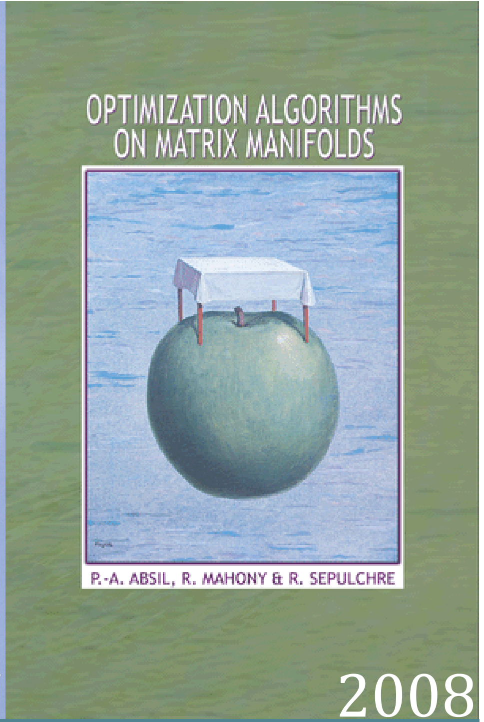


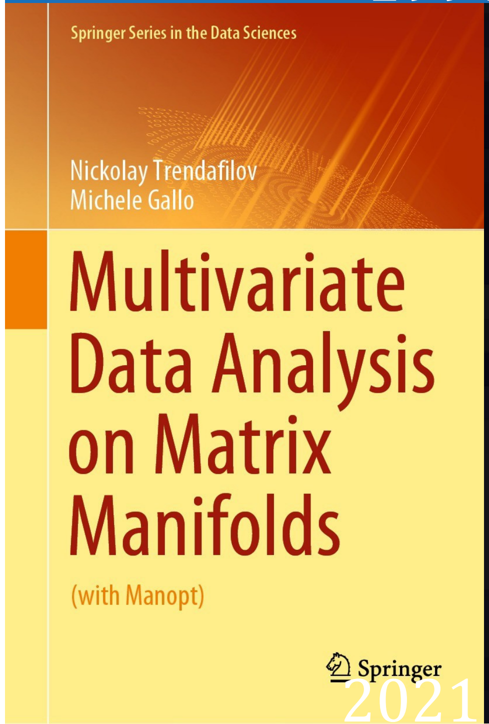
Becoming mainstream now.

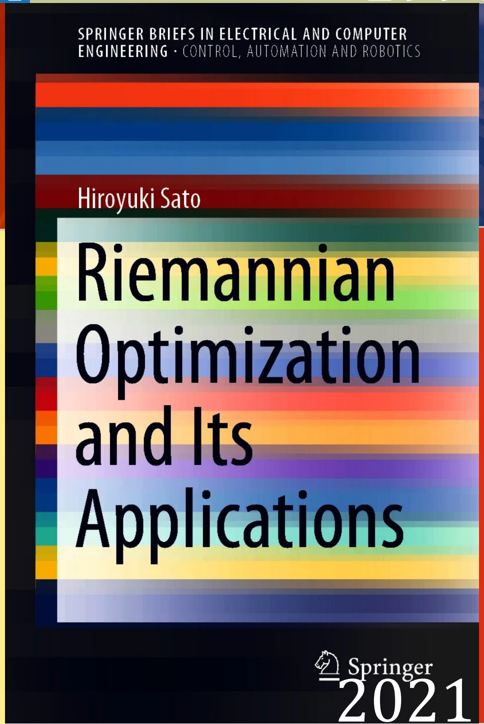

Communications and Control Engineering
Series Editor: Alberto Isidori · Jan H. van Schuppen
Eduardo D. Sontag · Manfred Thoma · Miroslav Krstic
Uwe Helmke · John B. Moore
R. Brockett Editors
Optimization and Dynamical Systems
1994

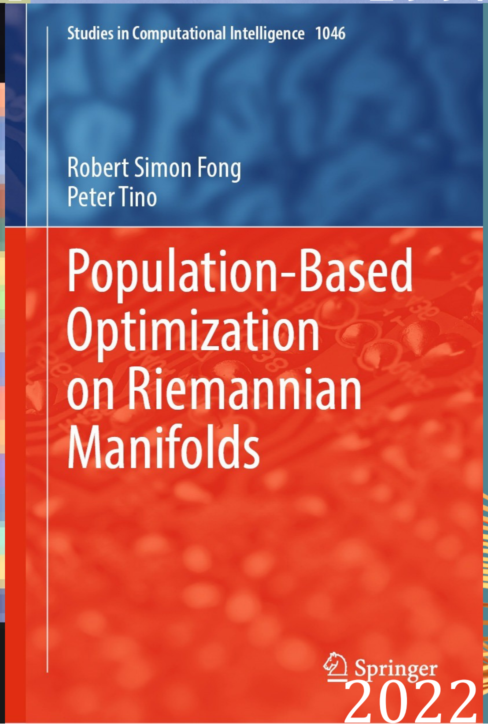

Mathematics and Its Applications
Constantin Udrişte
Convex Functions and Optimization Methods on Riemannian Manifolds
Springer-Science+Business Media B.V.
1994

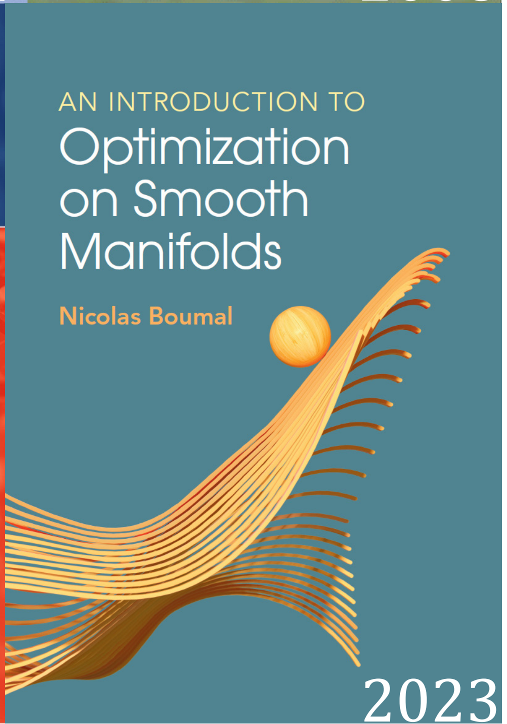

NONCONVEX OPTIMIZATION AND ITS APPLICATIONS
Smooth Nonlinear Optimization in R^n
Tamás Rapcsák
SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.
1997


OPTIMIZATION ALGORITHMS ON MATRIX MANIFOLDS
P.-A. ABSIL, R. MAHONY & R. SEPULCHRE
2008


Springer Series in the Data Sciences
Nickolay Trendafilov
Michele Gallo
Multivariate Data Analysis on Matrix Manifolds
(with Manopt)
Springer
2021


SPRINGER BRIEFS IN ELECTRICAL AND COMPUTER ENGINEERING · CONTROL, AUTOMATION AND ROBOTICS
Hiroyuki Sato
Riemannian Optimization and Its Applications
Springer
2021


Studies in Computational Intelligence 1046
Robert Simon Fong
Peter Tino
Population-Based Optimization on Riemannian Manifolds
Springer
2022


AN INTRODUCTION TO
Optimization on Smooth Manifolds
Nicolas Boumal
2023

Software, book, lectures, slides

Manopt software packages

manopt.org

Matlab with Bamdev Mishra, P.-A. Absil, R. Sepulchre++

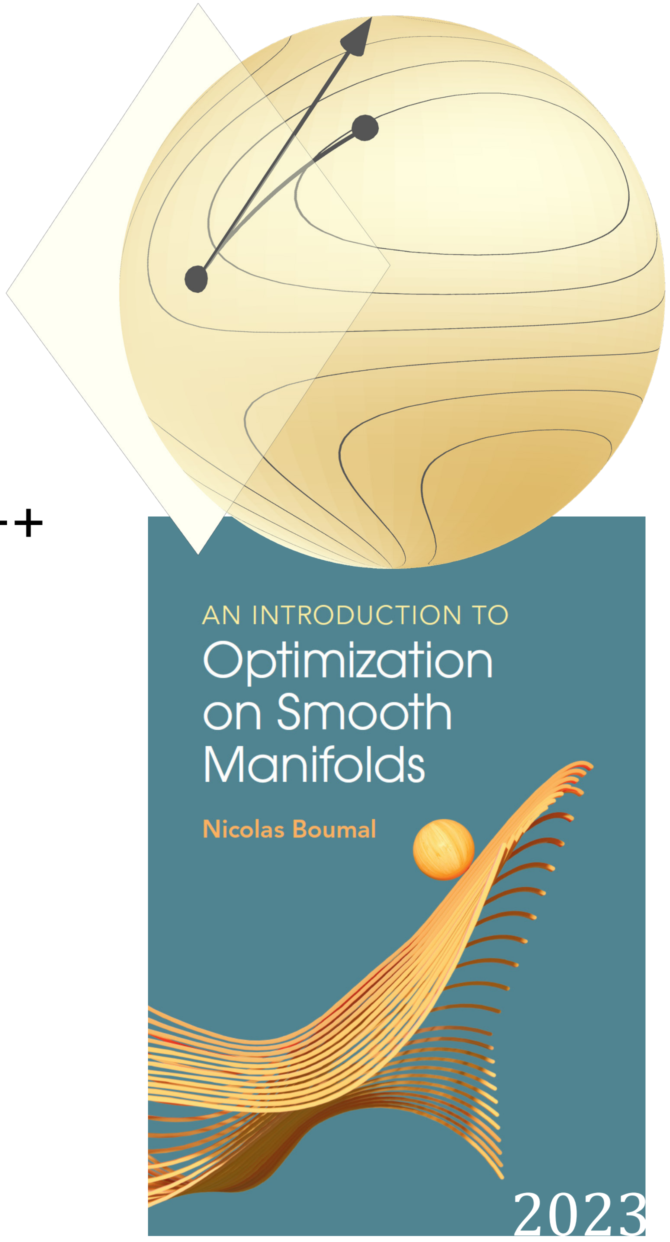
Julia by Ronny Bergmann++

Python by James Townsend, Niklas Koep
and Sebastian Weichwald++

Book (pdf, lecture material, videos) and these slides

nicolasboumal.net/book

nicolasboumal.net/SIAMOP23



Many thanks to Cambridge University Press, who agreed for me to keep the preprint freely available online.

How do manifolds arise in optimization?

Linear spaces

Symmetry

Orthonormality

Lifts/parameterizations

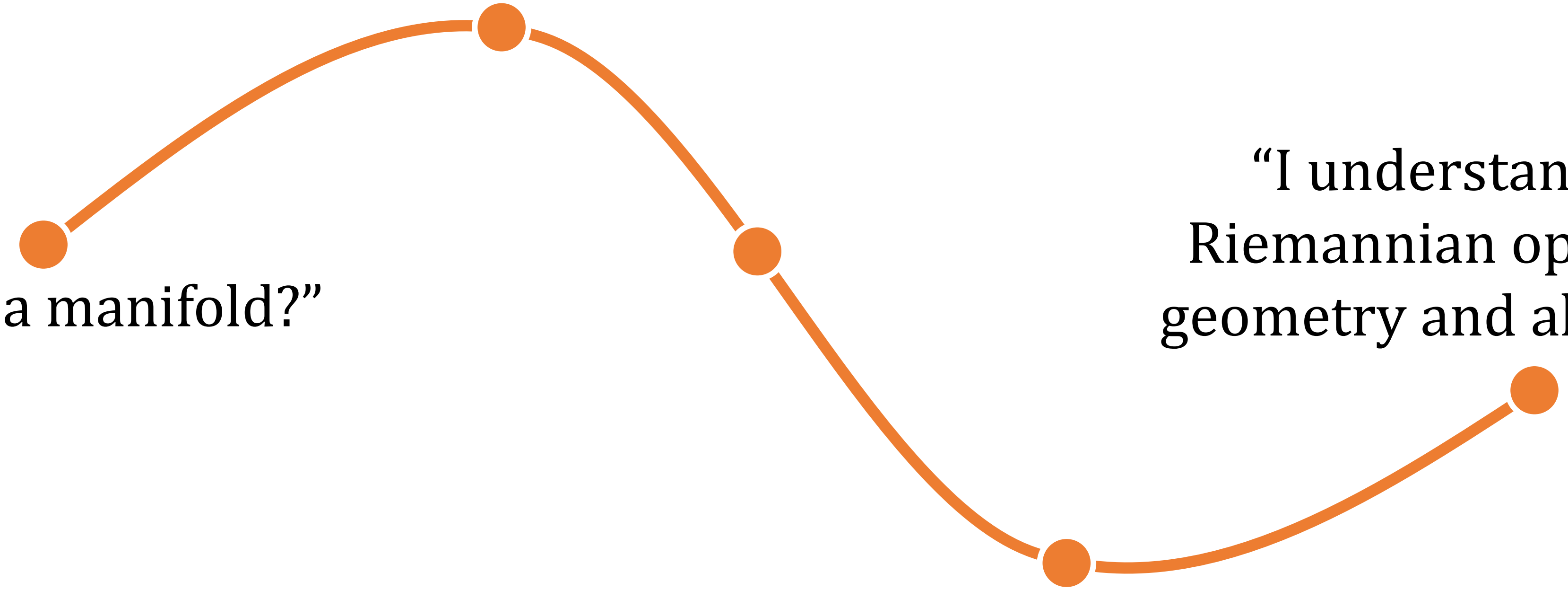
arXiv:2207.03512, with Eitan Levin & Joe Kileel

Positivity

Rank

Products

The goal for this tutorial



“What’s a manifold?”

“I understand basics of Riemannian optimization geometry and algorithms.”

Main effort: building differential geometry in ~ 2 hours.

Think of it as a technically precise bird’s-eye view, focused on intuition.

What do we need?

$$\min_x f(x)$$

	Euclidean optimization	Riemannian optimization
Basic step:	$x_{k+1} = x_k + s_k$	$x_{k+1} = R_{x_k}(s_k)$
Gradient descent:	$s_k = -\alpha_k \mathrm{grad} f(x_k)$	same, with Riemannian gradient
Newton's method:	$\mathrm{Hess} f(x_k)[s_k] = -\mathrm{grad} f(x_k)$	and Riemannian Hessian.

(Fancier algorithms involve more substantial differences, especially in analysis.)

Today, we build the following tools, from the ground up.

Hess f

Connections $\nabla, \frac{\mathrm{D}}{\mathrm{d}t}$

grad f

Riemannian metric $\langle u, v \rangle_x$

Vector fields

Retractions

$\mathrm{D}F(x)[v]$

Tangent bundle $\mathrm{T}\mathcal{M}$

What is a smooth function?

What is a tangent vector?

What is a smooth set?

Focus on embedded submanifolds of linear spaces.

What is a manifold? Take zero: words

Let $\mathcal{E}$ be a linear space (say, $\mathcal{E} = \mathbf{R}^d$).

A subset $\mathcal{M}$ of that linear space is a smooth manifold if,

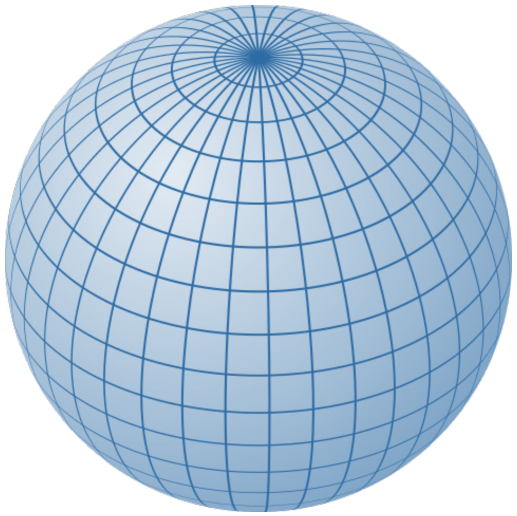
for each point $x \in \mathcal{M}$,

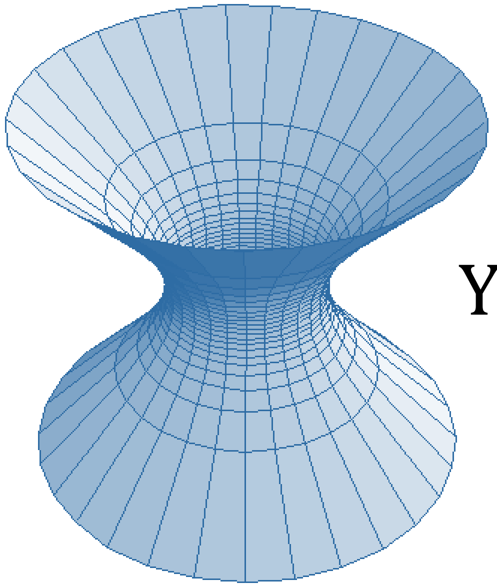
if we zoom very close,

it's hard to tell whether $\mathcal{M}$ is linear.

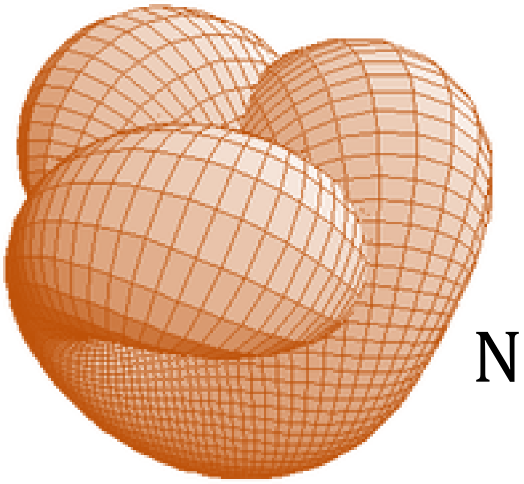
What is a manifold? Take one: pictures

Yes

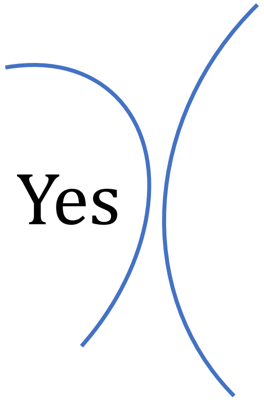




Yes



No



Yes

No

What is a manifold? Take two: examples

Linear spaces: $\mathbf{R}^n, \mathbf{R}^{m\times n}, \ldots$

Stiefel manifold: $\{X \in \mathbf{R}^{n\times p} : X^\top X = I_p\}$

Rotation group: $\{X \in \mathbf{R}^{n\times n} : X^\top X = I_n \text{ and } \det(X) = +1\}$

Fixed-rank matrices: $\{X \in \mathbf{R}^{m\times n} : \operatorname{rank}(X) = r\}$

Grassmann manifold: $\{X \in \mathbf{R}^{n\times n} : X = X^\top, X^2 = X, \operatorname{Tr}(X) = p\}$

Positive definite cone: $\{X \in \mathbf{R}^{n\times n} : X = X^\top \text{ and } X \succ 0\}$

Hyperbolic space: $\{x \in \mathbf{R}^{n+1} : x_0^2 = 1 + x_1^2 + \cdots + x_n^2 \text{ and } x_0 > 0\}$

…

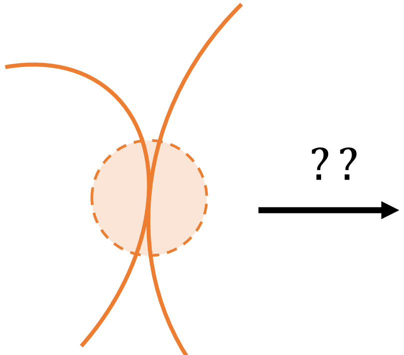
And products: if $\mathcal{M}_1, \mathcal{M}_2$ are manifolds, then $\mathcal{M}_1 \times \mathcal{M}_2$ is too.

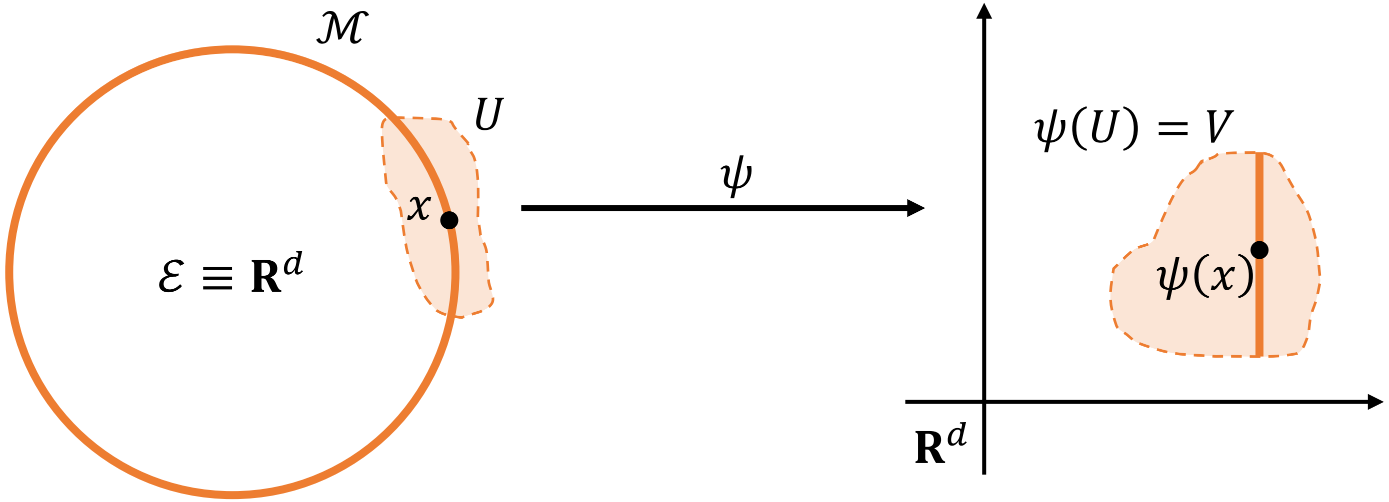
What is a manifold? Take three: math

A subset $\mathcal{M}$ of a linear space $\mathcal{E}$ of dimension $\dim \mathcal{E} = d$ is a smooth embedded submanifold of dimension $\dim \mathcal{M} = n$ if:

For all $x \in \mathcal{M}$, there exists a neighborhood U of x in $\mathcal{E}$, an open set $V \subseteq \mathbf{R}^d$ and a diffeomorphism $\psi: U \to V$ such that $\psi(U \cap \mathcal{M}) = V \cap E$ where E is a linear subspace of dimension n.

We call $\mathcal{E}$ the embedding space.





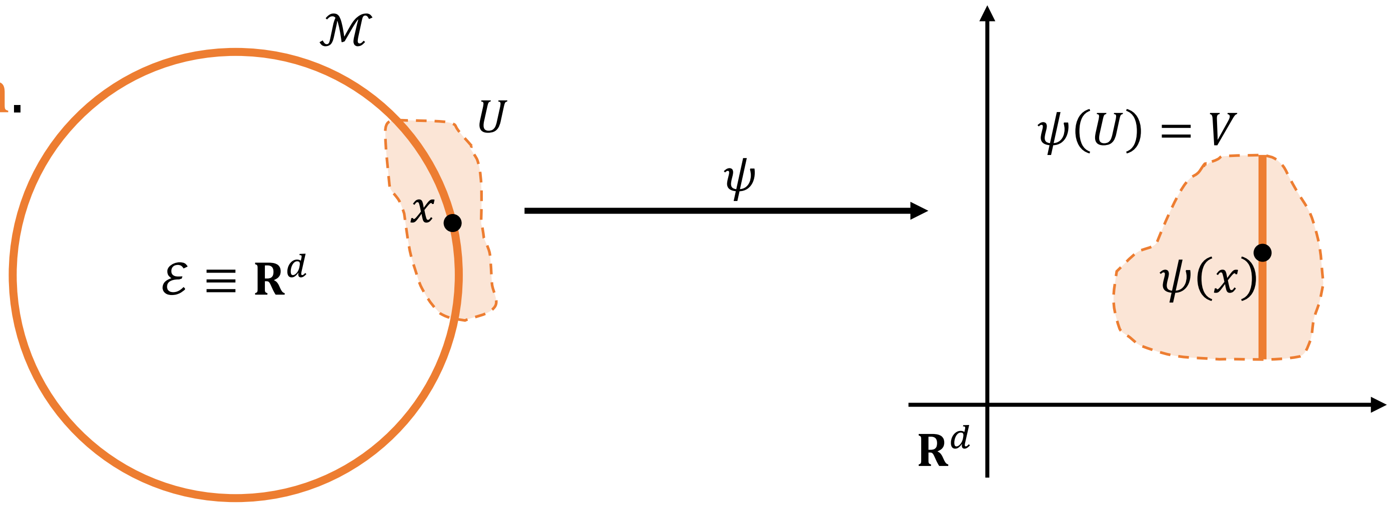
What is a manifold? Take four: math (bis)

A subset $\mathcal{M}$ of a linear space $\mathcal{E}$ of dimension $\dim \mathcal{E} = d$ is
a smooth embedded submanifold of dimension $\dim \mathcal{M} = n$ if:

For all $x \in \mathcal{M}$, there exists a neighborhood U of x in $\mathcal{E}$ and a smooth function $h: U \to \mathbf{R}^{d-n}$ such that $\mathcal{M} \cap U = \{y \in U: h(y) = 0\}$ and $\mathrm{D}h(x)$ has full rank.

We call h a local defining function.

In words: $\mathcal{M}$ is locally defined by smooth, independent equations.

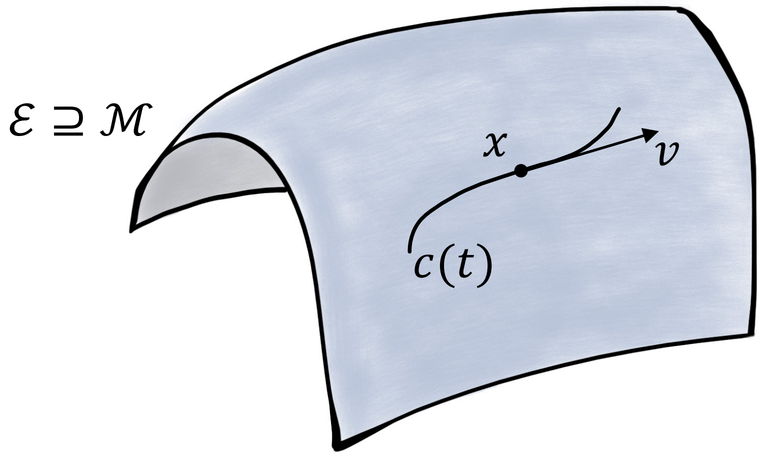


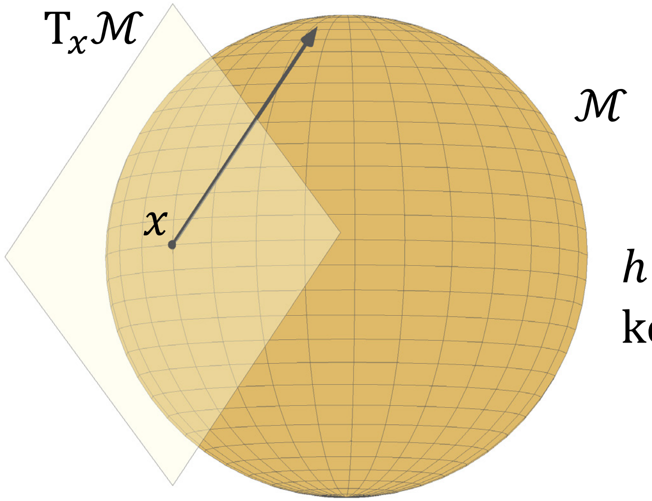
Tangent vectors of $\mathcal{M}$ embedded in $\mathcal{E}$

A tangent vector at x is the velocity $c'(0) = \lim_{t \to 0} \frac{c(t) - c(0)}{t}$ of a curve $c \colon \mathbf{R} \to \mathcal{M}$ with $c(0) = x$.

The tangent space $\mathrm{T}_x\mathcal{M}$ is the set of all tangent vectors of $\mathcal{M}$ at x.
It is a linear subspace of $\mathcal{E}$ of the same dimension as $\mathcal{M}$.

If $\mathcal{M} = \{x : h(x) = 0\}$ with $h \colon \mathcal{E} \to \mathbf{R}^k$ smooth and rank $\mathrm{D}h(x) = k$, then $\mathrm{T}_x\mathcal{M} = \ker \mathrm{D}h(x)$.





$h(x) = x^\top x - 1 = 0$
$\ker \mathrm{D}h(x) = \{v : x^\top v = 0\}$

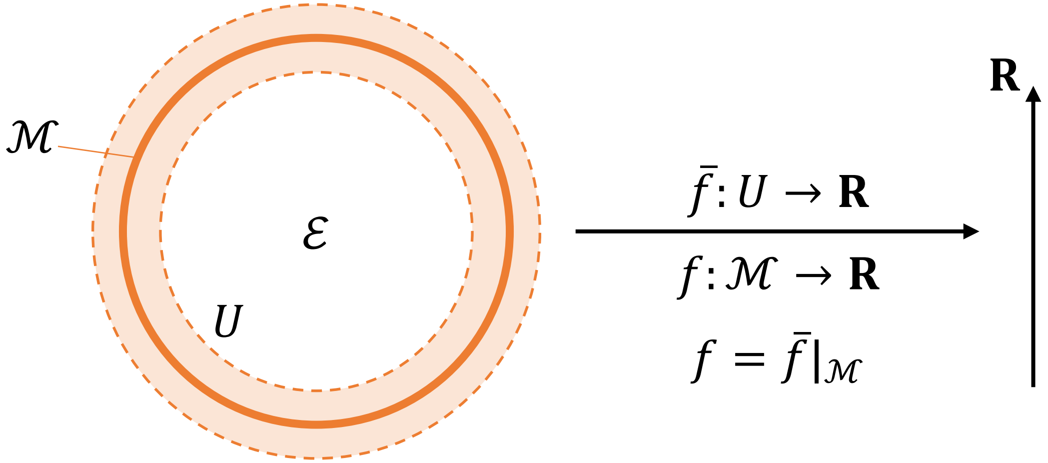
Smooth maps on/to manifolds

Let $\mathcal{M}, \mathcal{M}'$ be (smooth, embedded) submanifolds of linear spaces $\mathcal{E}, \mathcal{E}'$.

A map $F: \mathcal{M} \to \mathcal{M}'$ is smooth if it has a smooth extension, i.e., if there exists a neighborhood U of $\mathcal{M}$ in $\mathcal{E}$ and a smooth map $\bar{F}: U \to \mathcal{E}'$ such that $F = \bar{F}|_{\mathcal{M}}$.

Example: a cost function $f: \mathcal{M} \to \mathbf{R}$ is smooth if it is the restriction of a smooth $\bar{f}: U \to \mathbf{R}$.

Composition preserves smoothness.



Differential of a smooth map $F: \mathcal{M} \to \mathcal{M}'$

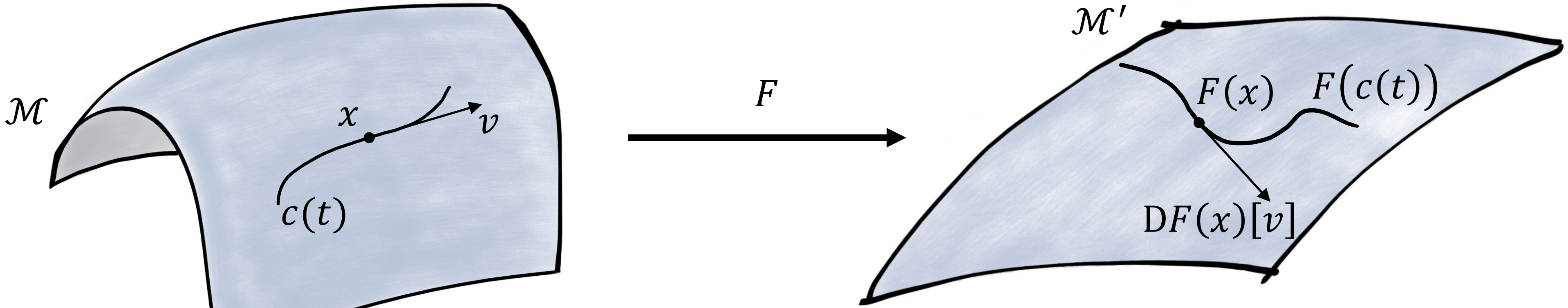
The differential of F at x is the map $\mathrm{D}F(x): \mathrm{T}_x\mathcal{M} \to \mathrm{T}_{F(x)}\mathcal{M}'$ defined by:

$$\mathrm{D}F(x)[v] = (F \circ c)'(0) = \lim_{t \to 0} \frac{F\big(c(t)\big) - F(x)}{t}$$

where $c: \mathbf{R} \to \mathcal{M}$ satisfies $c(0) = x$ and $c'(0) = v$.

Claim: $\mathrm{D}F(x)$ is well defined and linear, and we have a chain rule.

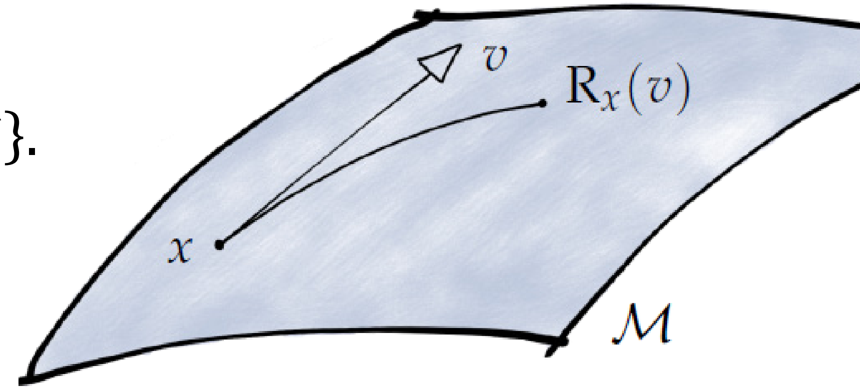
If $\bar{F}$ is a smooth extension of F, then $\mathrm{D}F(x) = \mathrm{D}\bar{F}(x)|_{\mathrm{T}_x\mathcal{M}}$.



Retractions: moving around on $\mathcal{M}$

The tangent bundle is the set

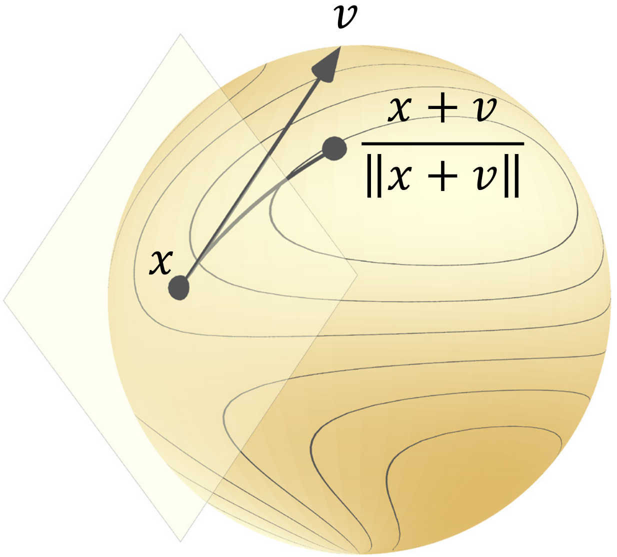
$$\mathrm{T}\mathcal{M} = \{(x, v) : x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}.$$



A retraction is a map $R : \mathrm{T}\mathcal{M} \to \mathcal{M} : (x, v) \mapsto R_x(v)$
such that each curve

$$c(t) = R_x(tv)$$

satisfies $c(0) = x$ and $c'(0) = v$.

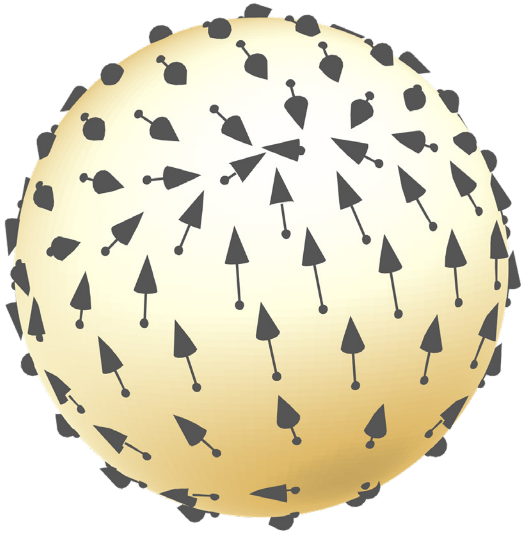


E.g., metric projection: $R_x(v)$ is the projection of $x + v$ to $\mathcal{M}$.

$\mathcal{M} = \mathbf{R}^n$: $R_x(v) = x + v$; $\mathcal{M} = \{x : \|x\| = 1\}$: $R_x(v) = \frac{x+v}{\|x+v\|}$;

$\mathcal{M} = \{X : \mathrm{rank}(X) = r\}$: $R_X(V) = \mathrm{SVD}_r(X + V)$.

Riemannian manifolds



Each tangent space $\mathrm{T}_x\mathcal{M}$ is a linear space.
Endow each one with an inner product: $\langle u, v \rangle_x$ for $u, v \in \mathrm{T}_x\mathcal{M}$.

A vector field is a map $V: \mathcal{M} \to \mathrm{T}\mathcal{M}$ such that $V(x)$ is tangent at x for all x.

We say the inner products $\langle \cdot, \cdot \rangle_x$ vary smoothly with x if $x \mapsto \langle U(x), V(x) \rangle_x$ is smooth for all smooth vector fields U, V.

If the inner products vary smoothly with x, they form a Riemannian metric.

A Riemannian manifold is a smooth manifold with a Riemannian metric.

Riemannian structure and optimization

A Riemannian manifold is a smooth manifold with a smoothly varying choice of inner product on each tangent space.

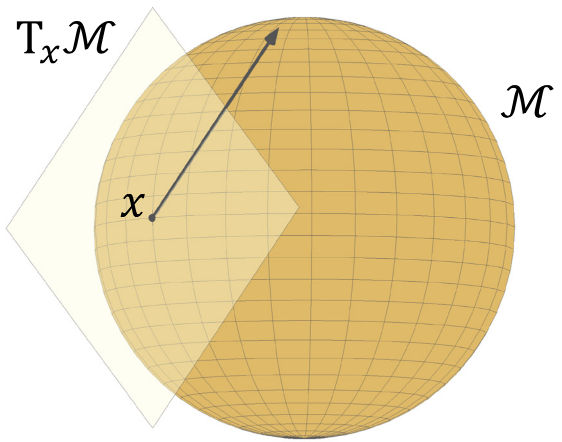
A manifold can be endowed with many different Riemannian structures.

A problem $\min_{x \in \mathcal{M}} f(x)$ is defined independently of any Riemannian structure.

We *choose* a metric for algorithmic purposes. Akin to preconditioning.

Riemannian submanifolds

Let the embedding space of $\mathcal{M}$ be a Euclidean space $\mathcal{E}$ with metric $\langle\cdot,\cdot\rangle$.
For example: $\mathcal{E} = \mathbf{R}^d$ and $\langle u, v\rangle = u^\top v$ for all $u, v \in \mathbf{R}^d$.



A convenient choice of Riemannian structure for $\mathcal{M}$ is to let:

$$\langle u, v\rangle_x = \langle u, v\rangle.$$

This is well defined because $u, v \in \mathrm{T}_x\mathcal{M}$ are, in particular, elements of $\mathcal{E}$.

This is a Riemannian metric. With it, $\mathcal{M}$ is a Riemannian submanifold of $\mathcal{E}$.

!! A Riemannian submanifold is *not* just a submanifold that is Riemannian !!

$$\langle \mathrm{grad}\bar{f}(x), v\rangle = \mathrm{D}\bar{f}(x)[v] = \lim_{t\to 0}\frac{\bar{f}(x+tv)-\bar{f}(x)}{t}$$

(Reminders for $\bar{f}\colon \mathbf{R}^d \to \mathbf{R}$.)

Riemannian gradients

The Riemannian gradient of a smooth $f\colon \mathcal{M} \to \mathbf{R}$ is the vector field $\mathrm{grad}f$ defined by:

$$\forall (x,v) \in \mathrm{T}\mathcal{M}, \qquad \langle \mathrm{grad}f(x), v\rangle_x = \mathrm{D}f(x)[v].$$

Claim: $\mathrm{grad}f$ is a well-defined smooth vector field.

If $\mathcal{M}$ is a Riemannian submanifold of a Euclidean space $\mathcal{E}$, then

$$\mathrm{grad}f(x) = \mathrm{Proj}_x\left(\mathrm{grad}\bar{f}(x)\right),$$

where Proj_x is the orthogonal projector from $\mathcal{E}$ to $\mathrm{T}_x\mathcal{M}$ and $\bar{f}$ is a smooth extension of f.

We're all set for gradient descent

$$x_{k+1} = R_{x_k}\big(-\alpha_k \mathrm{grad} f(x_k)\big)$$

How does $f(x_{k+1})$ compare to $f(x_k)$?

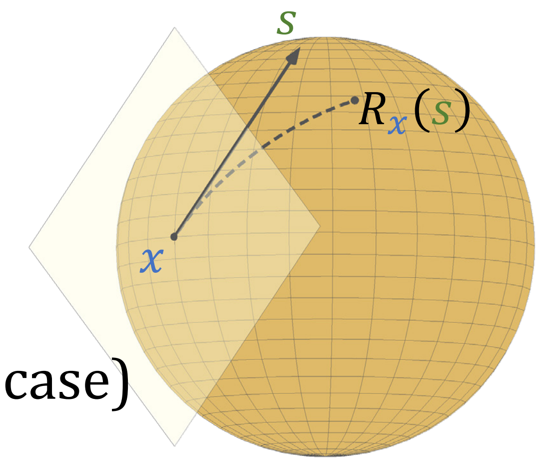
Consider a Taylor expansion of the pullback $f \circ R_x \colon \mathrm{T}_x\mathcal{M} \to \mathbf{R}$:

$$f\big(R_x(s)\big) = f(x) + \langle \mathrm{grad} f(x), s \rangle_x + O\big(\|s\|_x^2\big)$$

Gradient descent on $\mathcal{M}$

A1 $f(x) \geq f_{\text{low}}$ for all $x \in \mathcal{M}$

A2 $f\big(R_x(s)\big) \leq f(x) + \langle s, \operatorname{grad} f(x)\rangle_x + \frac{L}{2}\|s\|_x^2$



Algorithm: $x_{k+1} = R_{x_k}\left(-\frac{1}{L}\operatorname{grad} f(x_k)\right)$

Complexity: $\left[\min_{k<K}\|\operatorname{grad} f(x_k)\|_{x_k}\right] \leq \sqrt{\frac{2L(f(x_0)-f_{\text{low}})}{K}}$ (same as Euclidean case)

$$\textbf{A2} \Rightarrow f(x_{k+1}) \leq f(x_k) - \frac{1}{L}\|\operatorname{grad} f(x_k)\|_{x_k}^2 + \frac{1}{2L}\|\operatorname{grad} f(x_k)\|_{x_k}^2$$

$$\Rightarrow f(x_k) - f(x_{k+1}) \geq \frac{1}{2L}\|\operatorname{grad} f(x_k)\|_{x_k}^2$$

$$\textbf{A1} \Rightarrow f(x_0) - f_{\text{low}} \geq f(x_0) - f(x_K) = \sum_{k=0}^{K-1} f(x_k) - f(x_{k+1}) \geq \frac{K}{2L}\min_{k<K}\|\operatorname{grad} f(x_k)\|_{x_k}^2$$

$$\langle \mathrm{grad}\bar{f}(x), v\rangle = \mathrm{D}\bar{f}(x)[v] = \lim_{t\to 0} \frac{\bar{f}(x+tv) - \bar{f}(x)}{t}$$

$$\mathrm{Hess}\bar{f}(x)[v] = \mathrm{D}\big(\mathrm{grad}\bar{f}\big)(x)[v] = \lim_{t\to 0} \frac{\mathrm{grad}\bar{f}(x+tv) - \mathrm{grad}\bar{f}(x)}{t}$$

(Reminders for $\bar{f}: \mathbf{R}^d \to \mathbf{R}$.)

Riemannian Hessians

The Riemannian Hessian of f at x should be a symmetric linear map describing gradient change: $\mathrm{Hess}f(x): \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}$.

Since $\mathrm{grad}f: \mathcal{M} \to \mathrm{T}\mathcal{M}$ is a smooth map, a natural first attempt is:

$$\mathrm{Hess}f(x)[v] \stackrel{?}{=} \mathrm{Dgrad}f(x)[v].$$

However, the rhs is not always in $\mathrm{T}_x\mathcal{M}$... We need a new derivative for vector fields.

Fundamental theorem of Riemannian geometry:

There exists a unique way to differentiate vector fields that has "desirable properties".
This Riemannian connection ∇ leads to the Riemannian Hessian

$$\mathrm{Hess}f(x)[v] = \nabla_v \mathrm{grad}f$$

being a symmetric map on $\mathrm{T}_x\mathcal{M}$.

$$\langle \mathrm{grad}\bar f(x), v\rangle = \mathrm{D}\bar f(x)[v] = \lim_{t\to 0} \frac{\bar f(x+tv) - \bar f(x)}{t}$$

$$\mathrm{Hess}\bar f(x)[v] = \mathrm{D}\big(\mathrm{grad}\bar f\big)(x)[v] = \lim_{t\to 0} \frac{\mathrm{grad}\bar f(x+tv) - \mathrm{grad}\bar f(x)}{t}$$

(Reminders for $\bar f\colon \mathbf{R}^d \to \mathbf{R}$.)

Riemannian Hessians

The Riemannian Hessian of f at x should be a symmetric linear map describing gradient change: $\mathrm{Hess} f(x)\colon \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}$.

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However, the rhs is not always in $\mathrm{T}_x\mathcal{M}$... We need a new derivative for vector fields.

If $\mathcal{M}$ is a Riemannian submanifold of Euclidean space, then:

$$\begin{aligned}\mathrm{Hess} f(x)[v] &= \mathrm{Proj}_x(\mathrm{D}\mathrm{grad} f(x)[v]) \\ &= \mathrm{Proj}_x\big(\mathrm{Hess}\bar f(x)[v]\big) + W\left(v, \mathrm{Proj}_x^{\perp}\left(\mathrm{grad}\bar f(x)\right)\right)\end{aligned}$$

where W is the Weingarten map of $\mathcal{M}$.

Example: Rayleigh quotient optimization

Compute the smallest eigenvalue of a symmetric matrix $A \in \mathbf{R}^{n \times n}$:

$$\min_{x \in \mathcal{M}} \ \tfrac{1}{2} x^\top A x \qquad \text{with} \qquad \mathcal{M} = \left\{ x \in \mathbf{R}^n : x^\top x = 1 \right\}$$

The cost function $f \colon \mathcal{M} \to \mathbf{R}$ is the restriction of the smooth function $\bar{f}(x) = \frac{1}{2} x^\top A x$ from $\mathbf{R}^n$ to $\mathcal{M}$.

Tangent spaces $\qquad \mathrm{T}_x\mathcal{M} = \left\{ v \in \mathbf{R}^n : x^\top v = 0 \right\}$.

Make $\mathcal{M}$ into a Riemannian submanifold of $\mathbf{R}^n$ with $\langle u, v \rangle = u^\top v$.

Projection to $\mathrm{T}_x\mathcal{M}$: $\qquad \mathrm{Proj}_x(z) = z - \left(x^\top z \right) x$.

Gradient of $\bar{f}$: $\qquad \mathrm{grad}\bar{f}(x) = Ax$.

Gradient of f: $\qquad \mathrm{grad}f(x) = \mathrm{Proj}_x\left(\mathrm{grad}\bar{f}(x) \right) = Ax - \left(x^\top A x \right) x$.

Differential of $\mathrm{grad}f$: $\qquad \mathrm{Dgrad}f(x)[v] = Av - \left(v^\top A x + x^\top A v \right) x - \left(x^\top A x \right) v$.

Hessian of f: $\qquad \mathrm{Hess}f(x)[v] = \mathrm{Proj}_x(\mathrm{Dgrad}f(x)[v]) = \mathrm{Proj}_x(Av) - \left(x^\top A x \right) v$.

The following are equivalent for $x \in \mathcal{M}$: x is a global minimizer; x is a unit-norm eigenvector of A for the least eigenvalue; $\mathrm{grad}f(x) = 0$ and $\mathrm{Hess}f(x) \succcurlyeq 0$.

Enough definitions.
Now let's use this tower.

Hess f

Connections $\nabla, \frac{\mathrm{D}}{\mathrm{d}t}$

grad f

Riemannian metric $\langle u, v \rangle_x$

Vector fields

Retractions

$\mathrm{D}F(x)[v]$

Tangent bundle $\mathrm{T}\mathcal{M}$

What is a smooth function?

What is a tangent vector?

What is a smooth set?

Example:

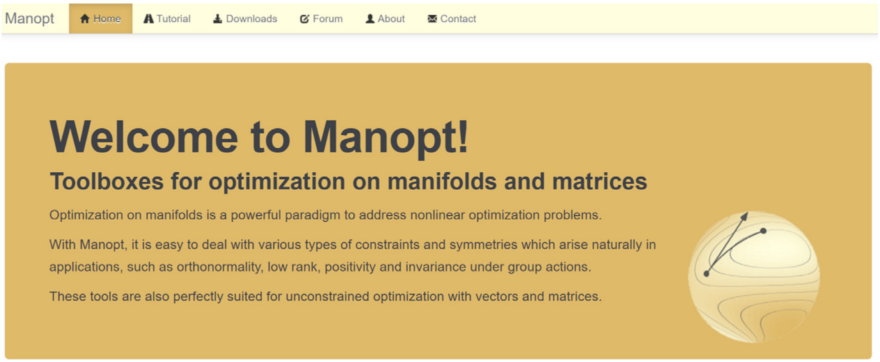
Max-Cut with Manopt

Full example: hands on with Manopt

Manopt is a family of toolboxes for Riemannian optimization.

Go to manopt.org for code, a tutorial, a forum, and a list of other software.

Github: github.com/NicolasBoumal/manopt



Matlab example for $\min_{\|x\|=1} x^\top A x$:

```
problem.M = spherefactory(n);
problem.cost = @(x) x'*A*x;
problem.egrad = @(x) 2*A*x;
x = trustregions(problem);
```

With Bamdev Mishra,
P.-A. Absil & R. Sepulchre

Lead by J. Townsend,
N. Koep & S. Weichwald

Lead by
Ronny Bergmann

What’s in a factory-produced manifold?

Example: stripped down and simplified `spherefactory`

```
function M = spherefactory(n)
    M.name = @() sprintf('Sphere S^%d', n-1);
    M.dim = @() n-1;
    M.inner = @(x, u, v) u'*v;
    M.norm = @(x, u) norm(u);
    M.dist = @(x, y) real(2*asin(.5*norm(x - y)));
```

```
M.exp = @exponential;
M.retr = @(x, u) (x+u)/norm(x+u);
M.invretr = @inverse_retraction;
M.log = @logarithm;
M.hash = @(x) ['z' hashmd5(x)];
M.rand = @() normalize(randn(n, 1));
));
```

```
function M = spherefactory(n)
  M.inner = @(x, u, v) u'*v;
  M.proj = @(x, u) u - x*(x'*u);
  M.egrad2rgrad = M.proj;
  M.ehess2rhess = @(x, egrad, ehess, u) ...
                  M.proj(x, ehess - (x'*egrad)*u);
  M.retr = @(x, u) (x+u)/norm(x+u);
```

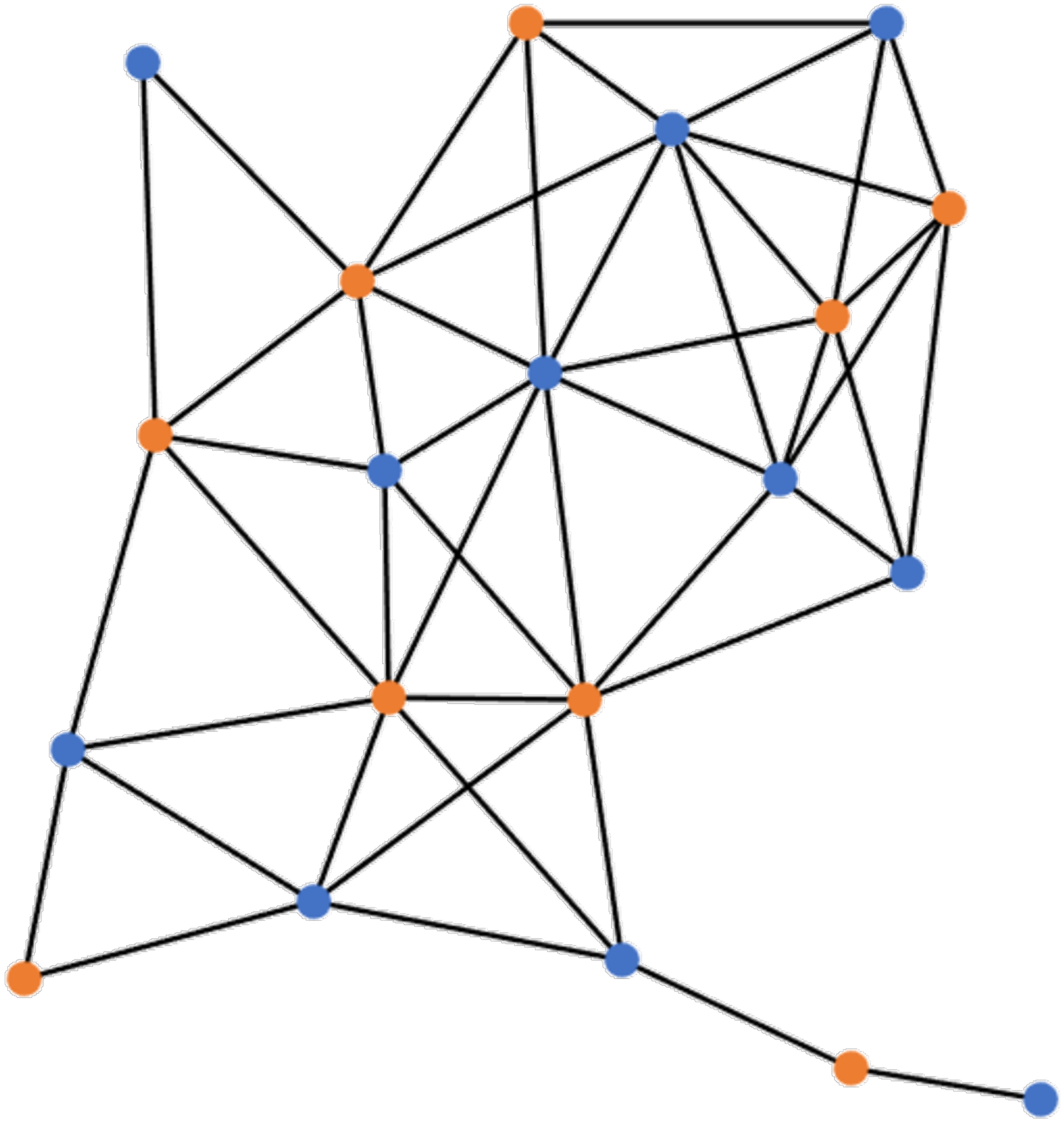
Max-Cut

Input:

An undirected graph.

Output:

Vertex labels ($+1$, -1) so that as many edges as possible connect different labels.



Max-Cut

Input:

An undirected graph:

adjacency matrix A.

Output:

Vertex labels $x_i \in \{+1, -1\}$

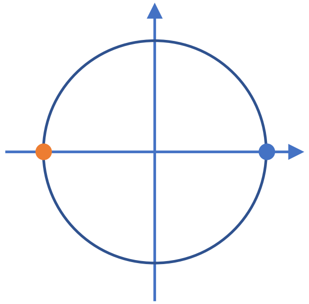
so that as many edges

as possible connect

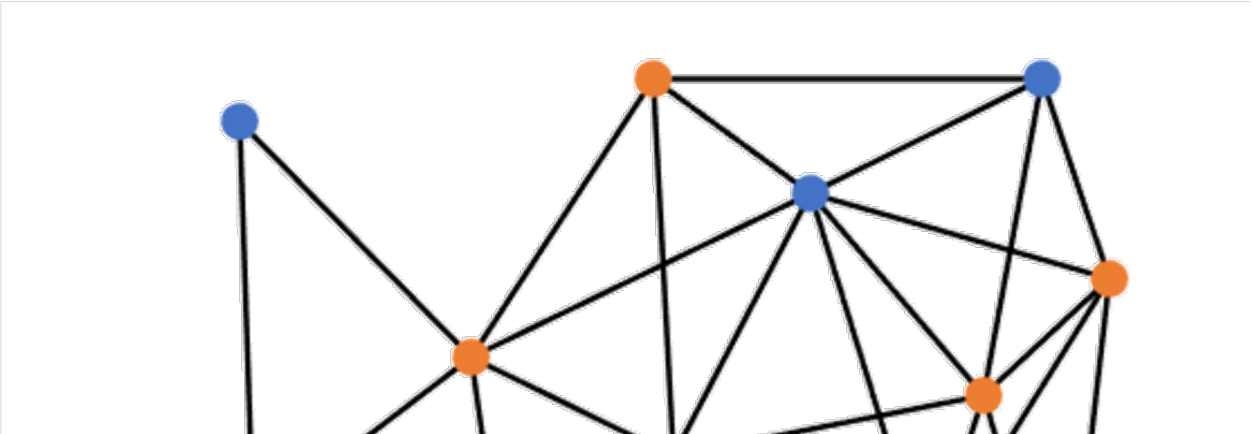
different labels.

$$\min_{x_1,\ldots,x_n} \sum_{ij} a_{ij} x_i x_j \quad \text{s.t.} \quad x_i \in \{\pm 1\}$$

Time-tested relaxation:



Let x_i be unit-norm in $\mathbf{R}^p$.



Max-Cut via low-rank relaxation in Manopt

With adjacency matrix $A \in \mathbf{R}^{n \times n}$, want:

$$\min_{x_1,\ldots,x_n \in \mathbf{R}^p} \sum_{ij} a_{ij} x_i^\top x_j \quad \text{s.t.} \quad \|x_i\| = 1 \ \forall i$$

The manifold is a product of n spheres:

$$\mathcal{M} = \{x \in \mathbf{R}^p : \|x\| = 1\}^n$$

$$\equiv \left\{X \in \mathbf{R}^{p \times n} : \left\|X_{:,i}\right\| = 1 \ \forall i\right\}$$

Called the oblique manifold.

```
data = load('graph20.mat');
A = data.A; n = data.n;

p = 3;
problem.M = obliquefactory(p, n);
problem.cost = @(X) sum((X*A) .* X, 'all');
problem.egrad = @(X) 2*X*A;
problem.ehess = @(X, Xdot) 2*Xdot*A;

X = trustregions(problem);

s = sign(X'*randn(p, 1)); %rand round
```

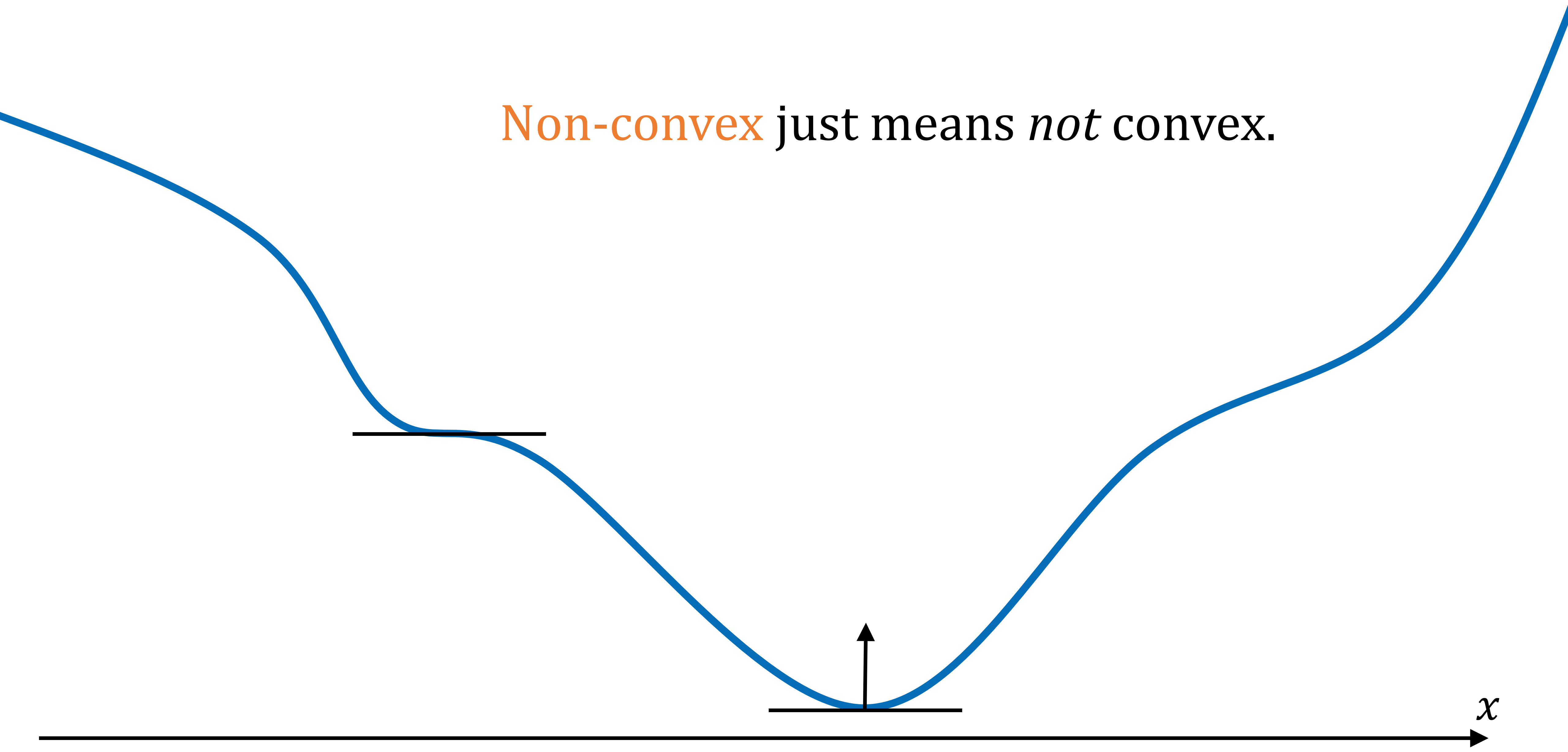
Active research directions by many

- More algorithms: nonsmooth, stochastic, parallel, randomized, ...
- Constrained optimization on manifolds
- Applications, old and new
- Complexity (upper and lower bounds, acceleration)
- Role of curvature
- Geodesic convexity
- Solution tracking (homotopy, continuation), bilevel, min-max
- Infeasible methods ("off-the-manifold", still using the structure)
- Broader generalizations: boundary, varieties, lift to smooth manifold, ...
- Benign non-convexity

"… in fact, the great watershed in optimization isn't between linearity and nonlinearity, but ***convexity*** *and* ***non-convexity****."*

R. T. Rockafellar, in SIAM Review, 1993

Non-convex just means *not* convex.



*"... in fact, the great watershed in optimization isn't between linearity and nonlinearity, but **convexity** and **non-convexity**."*

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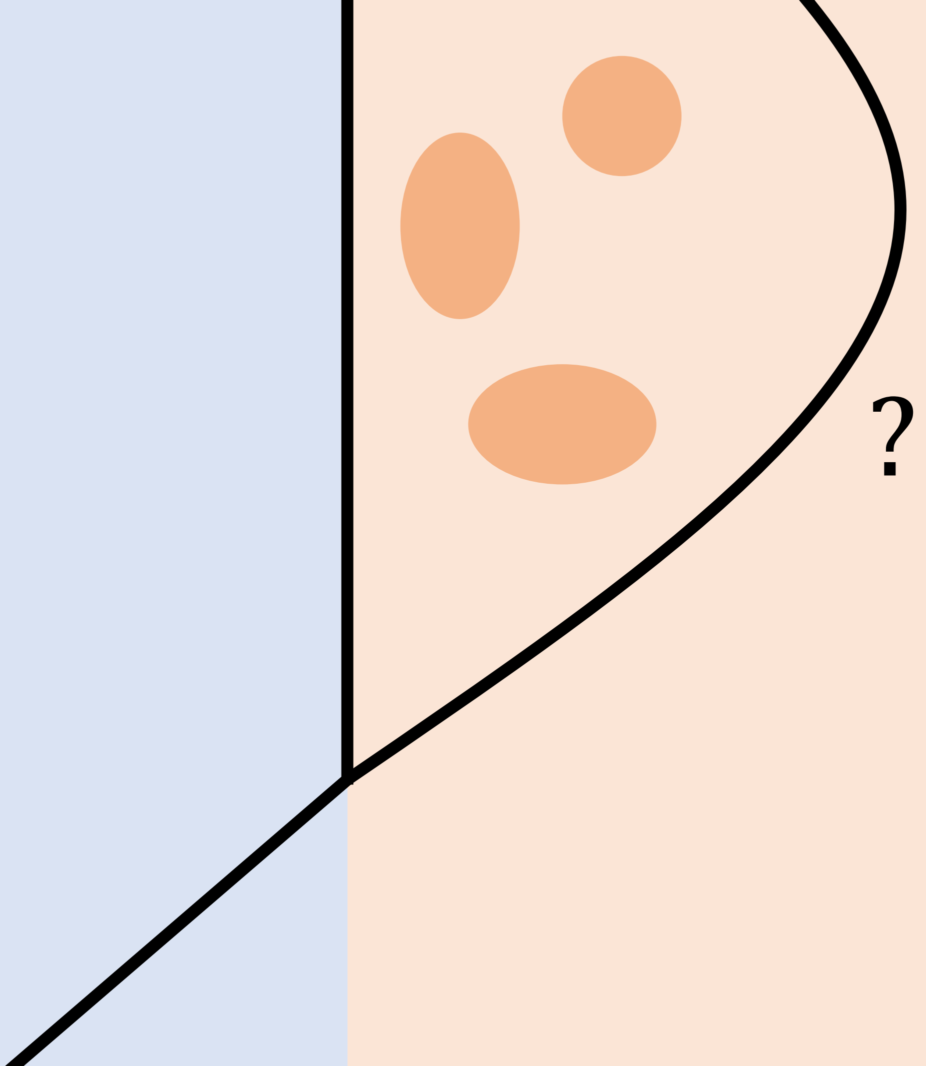


Pockets of benign non-convexity: Ju Sun's list

https://sunju.org/research/nonconvex, ~900 papers in March 2021; categories:

Matrix Completion/Sensing

Tensor Recovery/Decomposition & Hidden Variable Models

Phase Retrieval

Dictionary Learning

Deep Learning

Sparse Vectors in Linear Subspaces

Nonnegative/Sparse Principal Component Analysis

Mixed Linear Regression

Blind Deconvolution/Calibration

Super Resolution

Synchronization Problems Community Detection

Joint Alignment

Numerical Linear Algebra

Bayesian Inference

Empirical Risk Minimization & Shallow Networks

System Identification

Burer-Monteiro Style Decomposition Algorithms

Generic Structured Problems

Nonconvex Feasibility Problems

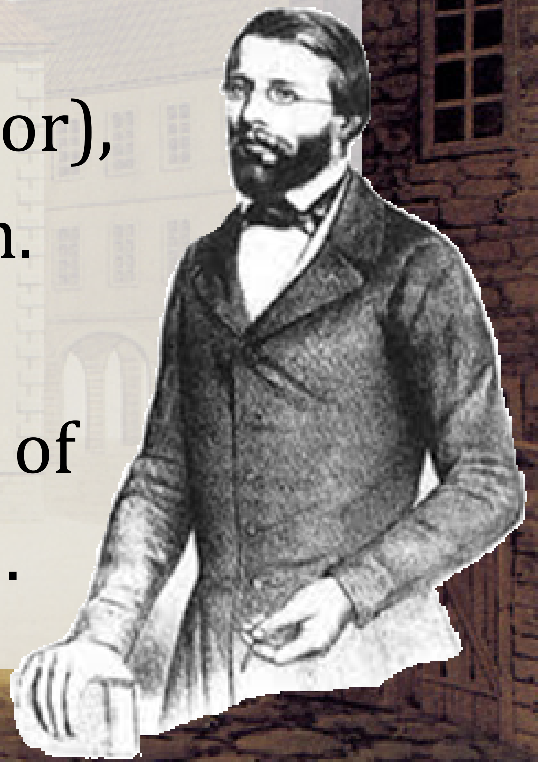
Separable Nonnegative Factorization (NMF)

Back in Göttingen...

If Riemann didn't invent his geometry to pick Netflix movies, then why did he?

His motivation was to extend the work of Gauss (his advisor), to understand **curvature** in spaces of arbitrary dimension.

Bit by bit, the community is building some understanding of the effect curvature has in optimization. To be continued…



Software, book, lectur

Saturday, June 3

MS311
Riemannian Optimization - Part III of III

3:15 PM - 4:45 PM
Room: Redwood B, 2nd floor

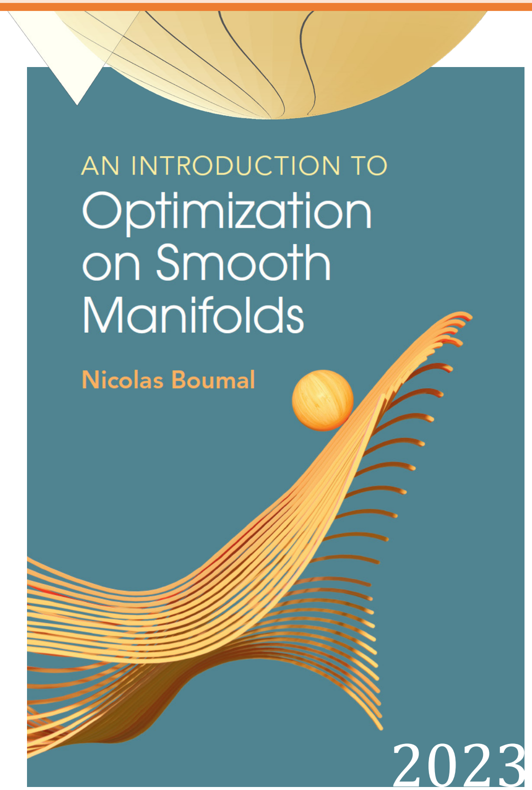
Manopt software packages

manopt.org

Matlab with Bamdev Mishra, P.-A. Absil, R. Sepulchre++

Julia by Ronny Bergmann++

Python by James Townsend, Niklas Koep and Sebastian Weichwald++



Book (pdf, lecture material, videos) and these slides

nicolasboumal.net/book

nicolasboumal.net/SIAMOP23

Many thanks to Cambridge University Press, who agreed for me to keep the preprint freely available online.